On the analyticity of the semigroups generated by the Hodge-Laplacian, and by the Stokes and Maxwell operators on Lipschitz subdomains of Riemannian manifolds

Marius Mitrea, Sylvie Monniaux

On the analyticity of the semigroups generated by the Hodge-Laplacian, and by the Stokes and Maxwell operators on Lipschitz subdomains of Riemannian manifolds

Marius Mitrea* and Sylvie Monniaux†

November 11, 2005

1 Introduction

This work is concerned with the study of the analyticity of the semigroup associated with the linearized, time-dependent Stokes system with Neumann boundary conditions

$$\begin{aligned} &\partial_t u - \Delta u + \nabla \pi = f, \quad \operatorname{div} u = 0 \ \text{ in } \ \Omega \times (0,T), \\ &\nu \times \operatorname{curl} u\big|_{\partial\Omega\times(0,T)} = 0, \quad \nu \cdot u\big|_{\partial\Omega\times(0,T)} = 0, \quad u\big|_{t=0} = u_o \text{ in } \Omega, \end{aligned} \tag{1.1}$$

as well as the parabolic Maxwell system equipped with a perfectly conduction wall condition, i.e.,

$$\begin{aligned} &\partial_t u + \operatorname{curl}\operatorname{curl} u = f, \quad \operatorname{div} u = 0 \ \text{ in } \ \Omega \times (0,T), \\ &\nu \times u\big|_{\partial\Omega\times(0,T)} = 0, \quad u\big|_{t=0} = u_o \text{ in } \Omega. \end{aligned} \tag{1.2}$$

In (1.1), u and π stand, respectively, for the velocity field and pressure of a fluid occupying a domain Ω, whereas, in (1.2), u denotes the magnetic field propagating

*Supported in part by the NSF grants DMS - 0400639 and DMS-FRG - 0456306
†Supported in part by the NSF grant DMS-FRG - 0456306 and a UMC Miller Scholarship
2000 *Mathematics Subject Classification.* Primary: 42B30, 46A16, 46E35; Secondary 42B20, 35J25
Key words and phrases. Hodge-Laplacian, Lipschitz domains, analytic semigroup, differential forms

inside of the domain Ω. In both cases, u_o denotes the initial datum, f is a given, divergence-free field and ν stands for the unit outward normal to $\partial\Omega$.

Systems such as (1.1) and (1.2) naturally arise in the process of linearizing some basic nonlinear evolution problems in mathematical physics, such as the Navier-Stokes equations and certain problems related to the Ginzburg-Landau model for superconductivity and magneto-hydrodynamics. A more detailed discussion in this regard can be found in the monographs [5] by T.G. Cowling, [9] by L.D. Landau and E.M. Lifshitz, [15] by M.E. Taylor and [6] by R. Dautray and J.L. Lions.

In a suitable L^2 context, the stationary versions of (1.1) and (1.2) have unique (finite energy) weak-solutions. This is most elegantly seen using the so-called $\{\mathcal{V}, \mathcal{H}, a\}$ formalism as in [6] which, among other things, also gives that the associated solution operators generate analytic semigroups in (appropriate subspaces of) L^2. Thus, the natural issue which arises here is whether the same is true in the L^p context, with $p \neq 2$. This aspect, which is particularly relevant when dealing with nonlinear versions of (1.1)-(1.2), is intimately connected with resolvent estimates for the stationary versions of (1.1)-(1.2). More specifically, for $\lambda \in \mathbb{C}$ and $1 < p < \infty$, consider the boundary-value problems

$$\begin{aligned}
&\lambda u - \Delta u + \nabla\pi = f \in L^p(\Omega)^3 \ \text{ with } \ \operatorname{div} f = 0, \quad \nu \cdot f\big|_{\partial\Omega} = 0,\\
&\operatorname{div} u = 0 \ \text{ in } \ \Omega,\\
&u, \operatorname{curl} u \in L^p(\Omega)^3, \ \pi \in W^{1,p}(\Omega),\\
&\nu \times \operatorname{curl} u\big|_{\partial\Omega} = 0, \quad \nu \cdot u\big|_{\partial\Omega} = 0,
\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}
&\lambda u - \operatorname{curl}\operatorname{curl} u = f \in L^p(\Omega)^3, \quad \operatorname{div} u = 0 \ \text{ in } \ \Omega,\\
&u \in L^p(\Omega)^3, \ \operatorname{curl} u \in L^p(\Omega)^3,\\
&\nu \times u\big|_{\partial\Omega} = 0.
\end{aligned} \tag{1.4}$$

In each case, the resolvent estimates alluded to before read

$$\|u\|_{L^p(\Omega)^3} \leq C(\Omega, p)\, |\lambda|\, \|f\|_{L^p(\Omega)^3} \tag{1.5}$$

uniformly in λ satisfying $|\arg(\lambda)| < \theta$ for some $\theta > 0$.

When the domain Ω has a sufficiently smooth boundary, such estimates are well-understood. The classical approach utilizes the fact that the boundary-value problems (1.3)-(1.4) are regular elliptic (cf., e.g., [15]) and the so-called "Agmon trick" (cf. [1]). See, for example, [10] where it is shown that (1.5) holds for each $p \in (1, \infty)$ if $\partial\Omega \in C^\infty$.

The nature of the problem at hand changes dramatically as $\partial\Omega$ becomes less regular. To illustrate this point, let us recall the following negative result from [4]. For each $p > 3$ there exists a bounded cone-like domain $\Omega \subset \mathbb{R}^3$ for which the resolvent estimate (1.5) fails in the case of the Stokes system equipped with a Dirichlet boundary condition, i.e. for

$$\begin{aligned}
&\lambda u - \Delta u + \nabla \pi = f \in L^p(\Omega)^3 \ \text{ with } \ \operatorname{div} f = 0, \quad \nu \cdot f\big|_{\partial\Omega} = 0, \\
&\operatorname{div} u = 0 \ \text{ in } \ \Omega, \quad u\big|_{\partial\Omega} = 0, \\
&u \in W^{1,p}(\Omega)^3, \ \pi \in L^p(\Omega).
\end{aligned} \tag{1.6}$$

The sharp nature of the aforementioned counterexample is also underscored by the following intriguing conjecture made by M. Taylor in [16].

Taylor's Conjecture. *For a given bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the Stokes operator (i.e. the solution operator) associated with (1.6) generates an analytic semigroup on L^p provided $3/2 - \varepsilon < p < 3 + \varepsilon$.*

The range of p's in the above conjecture is naturally dictated by the mapping properties of the Leray projection

$$P_p : L^p(\Omega)^3 \longrightarrow \{u \in L^p(\Omega)^3 : \operatorname{div} u = 0, \ \nu \cdot u = 0\}. \tag{1.7}$$

When $p = 2$ this is taken to be the canonical orthogonal projection and is obviously bounded, but the issue of whether this extends to a bounded operators in the context of (1.7) for other values of p is considerably more subtle. Indeed, it has been shown in [7] that for any bounded Lipschitz domain in $\mathbb{R}^n$ the operator (1.7) is bounded precisely for $3/2 - \varepsilon < p < 3 + \varepsilon$ for some $\varepsilon = \varepsilon(\Omega) > 0$ and that this result is sharp in the class of Lipschitz domains.

In this paper, we are able to prove the analogue of Taylor's conjecture for the Stokes system equipped with Neumann type boundary conditions. Our approach makes essential use of the recent progress in understanding Poisson type problems for the Hodge-Laplacian such as

$$\begin{aligned}
&-\Delta u = f \in L^p(\Omega)^3 \\
&\nu \cdot u\big|_{\partial\Omega} = 0, \quad \nu \times \operatorname{curl} u\big|_{\partial\Omega} = 0, \\
&u \in L^p(\Omega)^3, \ \operatorname{div} u \in W^{1,p}(\Omega), \ \operatorname{curl} u \in L^p(\Omega)^3.
\end{aligned} \tag{1.8}$$

Recently, it has been shown in [13] that, given a three-dimensional Lipschitz domain, there exist

$$1 \le p_\Omega < 2 < q_\Omega \le \infty, \qquad 1/p_\Omega + 1/q_\Omega = 1, \tag{1.9}$$

such that (1.8) is well-posed if and only if $p \in (p_\Omega, q_\Omega)$. Here, the index p_Ω can be further defined in terms of the critical exponents intervening in the (regular) Dirichlet and Neumann problems for the Laplace-Beltrami operator in Ω (as well as its complement), when optimal L^p estimates for the associated nontangential maximal function are sought. One feature of Ω which influences the size of p_Ω is the local oscillations of the unit conormal ν to $\partial\Omega$. In particular, $p_\Omega = 1$ (and, hence, $q_\Omega = \infty$) when ν belongs to the Sarason class of functions of vanishing mean oscillations (which is the case if, e.g., $\partial\Omega \in C^1$). Furthermore, for a Lipschitz polyhedron in the Euclidean setting, p_Ω can be estimated in terms of the dihedral angles involved.

Similar results have been proved earlier in [12] for dimensions greater than three, albeit the nature of the critical indices p_Ω, q_Ω is less clear (typically, they should be thought as small perturbations of 2).

One of our main results states that the Hodge-Laplacian (i.e., the solution operator associated with (1.8) generates an analytic semigroup in L^p for any $p \in (p_\Omega, q_\Omega)$. From this and the fact that the Leray projection commutes with the semigroup generated by the Hodge-Laplacian it is then possible to prove a similar conclusion for the Stokes and Maxwell operators acting on L^p (cf. Theorem 7.3 and Theorem 7.4 for precise statements). Thus, from this perspective, one key feature of our approach is to embed the Stokes (and Maxwell) system into a more general (and, ultimately, more manageable) elliptic problem, in a way which allows us to return to the original system by specializing the type of data allowed in the formulation of the problem.

We carry out this program in the context of differential forms on Lipschitz subdomains of a smooth, compact, boundaryless Riemannian manifold $\mathcal{M}$. This is both notational convenient and natural from a geometric point of view. It also allows for a more general setting than previously considered in the literature even in the case when $\partial\Omega \in C^1$ (semigroup methods for differential forms in the smooth context are discussed in, e.g., P.E. Conner's book [3]).

One distinctive aspect of our work is that, given the low regularity assumptions we make on the underlying domain Ω, we are forced to work with certain non-standard Sobolev-type spaces, which are well-adapted to the differential operators at hand (such as the exterior derivative operator d and its formal adjoint δ). In particular, issues such as boundary traces and embeddings become more delicate than in the standard theory.

The organization of the paper is as follows. In Section 2 we collect basic definitions and preliminary results, and introduce most of the notational conventions used throughout this work. The Hodge-Laplacian is reviewed in Section 3, along with the Stokes and Maxwell operators. In Section 4 we dwell on issues of regularity for differential forms in the domain of the Hodge-Laplacian. Here we record several key results, themselves corollaries of the work in [13] and [12]. Our strategy for determining all p's for which the Hodge-Laplacian generates an analytic semigroup in L^p is to start with $p = 2$ (when mere functional analysis will do) and then develop a bootstrap type argument which allows us to incrementally increase the value of the integrability exponent from p to $p^* := np/(n-1)$. This portion of our analysis involves a delicate inductive scheme which is executed in Section 5. Finally, the main results of the paper are stated and proved in Section 6, which deals with the issue of analyticity for the semigroup generated by the Hodge-Laplacian, as well as the Stokes and Maxwell operators.

2 Background material

In this section we review a number of basic definition and collect several known results which are going to be useful for us in the sequel.

2.1 Geometrical preliminaries

Let $\mathcal{M}$ be a smooth, compact, oriented, manifold of real dimension n, equipped with a smooth metric tensor, $g = \sum_{j,k} g_{jk} dx_j \otimes dx_k$. Denote by $T\mathcal{M}$ and $T^*\mathcal{M}$ the tangent and cotangent bundles to $\mathcal{M}$, respectively. Occasionally, we shall identify $T^*\mathcal{M} \equiv T\mathcal{M}$ canonically, via the metric. Set $\Lambda^\ell T\mathcal{M}$ for the ℓ-th exterior power of $T\mathcal{M}$. Sections in this vector bundle are ℓ-differential forms. The Hermitian structure on $T\mathcal{M}$ extends naturally to $T^*\mathcal{M} := \Lambda^1 T\mathcal{M}$ and, further, to $\Lambda^\ell T\mathcal{M}$. We denote by $\langle \cdot, \cdot \rangle$ the corresponding (pointwise) inner product. The volume form on $\mathcal{M}$, denoted in the sequel by ω, is the unique unitary, positively oriented, differential form of maximal degree on $\mathcal{M}$. In local coordinates, $\omega := [\det(g_{jk})]^{1/2} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$. In the sequel, we denote by dV the Borelian measure induced by ω on $\mathcal{M}$, i.e., $dV = [\det(g_{jk})]^{1/2} dx_1 dx_2 \ldots dx_n$.

Going further, we introduce the Hodge star operator as the unique vector bundle morphism $* : \Lambda^\ell T\mathcal{M} \to \Lambda^{n-\ell} T\mathcal{M}$ such that $u \wedge (*u) = |u|^2 \omega$ for each $u \in \Lambda^\ell T\mathcal{M}$. In particular, $\omega = *\,1$ and

$$u \wedge (*v) = \langle u, v \rangle \omega, \qquad \forall\, u \in \Lambda^\ell T\mathcal{M},\ \forall\, v \in \Lambda^\ell T\mathcal{M}. \tag{2.1}$$

The interior product between a 1-form ν and a ℓ-form u is then defined by

$$\nu \vee u := (-1)^{\ell(n+1)} * (\nu \wedge *u). \tag{2.2}$$

Let d stand for the (exterior) derivative operator and denote by δ its formal adjoint (with respect to the metric introduced above). For further reference some basic properties of these objects are summarized below.

Proposition 2.1. *For arbitrary* 1*-form* ν*,* ℓ*-form* u*,* $(n-\ell)$*-form* v*, and* $(\ell+1)$*-form* w*, the following are true:*

(1) $\langle u, *v\rangle = (-1)^{\ell(n-\ell)}\langle *u, v\rangle$ *and* $\langle *u, *v\rangle = \langle u, v\rangle$. *Also,* $**\,u = (-1)^{\ell(n-\ell)}\,u$;

(2) $\langle \nu \wedge u, w\rangle = \langle u, \nu \vee w\rangle$;

(3) $*(\nu \wedge u) = (-1)^{\ell}\nu \vee (*u)$ *and* $*(\nu \vee u) = (-1)^{\ell+1}\nu \wedge (*u)$;

(4) $*\delta = (-1)^{\ell} d*$, $\delta * = (-1)^{\ell+1} * d$, *and* $\delta = (-1)^{n(\ell+1)+1} * d*$ *on* ℓ*-forms.*

(5) $-(d\delta + \delta d) = \Delta$, *the Hodge-Laplacian on* $\mathcal{M}$.

Let Ω be a Lipschitz subdomain of $\mathcal{M}$. That is, $\partial\Omega$ can be described in appropriate local coordinates by means of graphs of Lipschitz functions. Then the unit conormal $\nu \in T^*\mathcal{M}$ is defined a.e., with respect to the surface measure $d\sigma$, on $\partial\Omega$. For any two sufficiently well-behaved differential forms (of compatible degrees) u, w we then have

$$\int_{\Omega} \langle du, w\rangle\, dV = \int_{\Omega} \langle u, \delta w\rangle\, dV + \int_{\partial\Omega} \langle u, \nu \vee w\rangle\, d\sigma. \tag{2.3}$$

2.2 Smoothness spaces

The Sobolev (or potential) class $L^p_\alpha(\mathcal{M})$, $1 < p < \infty$, $\alpha \in \mathbb{R}$, is obtained by lifting the Euclidean scale $L^p_\alpha(\mathbb{R}^n) := \{(I-\Delta)^{-\alpha/2} f : f \in L^p(\mathbb{R}^n)\}$ to $\mathcal{M}$ (via a C^∞ partition of unity and pull-back). For a Lipschitz subdomain Ω of $\mathcal{M}$, we denote by $L^p_\alpha(\Omega)$ the restriction of elements in $L^p_\alpha(\mathcal{M})$ to Ω, and set $L^p_\alpha(\Omega, \Lambda^\ell) = L^p_\alpha(\Omega) \otimes \Lambda^\ell T\mathcal{M}$, i.e. the collection of ℓ-forms with coefficients in $L^p_\alpha(\Omega)$. In particular, $L^p(\Omega, \Lambda^\ell)$ stands for the space of ℓ-differential forms with p-th power integrable coefficients in Ω.

Let us also note here that if $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$, then

$$\Big(L^p_s(\Omega, \Lambda^\ell)\Big)^* = L^{p'}_{-s}(\Omega, \Lambda^\ell), \qquad \forall\, s \in (-1 + 1/p, 1/p). \tag{2.4}$$

Next, denote by $L^p_1(\partial\Omega)$ the Sobolev space of functions in $L^p(\partial\Omega)$ with tangential gradients in $L^p(\partial\Omega)$, $1 < p < \infty$. Besov spaces on $\partial\Omega$ can then be introduced via real interpolation, i.e.

$$B^{p,q}_\theta(\partial\Omega) := \left(L^p(\partial\Omega), L^p_1(\partial\Omega)\right)_{\theta,q}, \quad \text{with } 0 < \theta < 1,\ 1 < p, q < \infty. \tag{2.5}$$

Finally, if $1 < p, q < \infty$ and $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, we define

$$B^{p,q}_{-s}(\partial\Omega) := \Big(B^{p',q'}_s(\partial\Omega)\Big)^*, \quad 0 < s < 1, \tag{2.6}$$

and, much as before, set $B^{p,q}_s(\partial\Omega, \Lambda^\ell) := B^{p,q}_s(\partial\Omega) \otimes \Lambda^\ell T\mathcal{M}$.

Recall (cf. [**8**]) that the trace operator

$$\mathrm{Tr} : L^p_\alpha(\Omega, \Lambda^\ell) \longrightarrow B^{p,p}_{\alpha - \frac{1}{p}}(\partial\Omega, \Lambda^\ell) \tag{2.7}$$

is well-defined, bounded and onto if $1 < p < \infty$ and $\frac{1}{p} < \alpha < 1 + \frac{1}{p}$. Furthermore, the trace operator has a bounded right inverse

$$\mathrm{Ex} : B^{p,p}_{\alpha - \frac{1}{p}}(\partial\Omega, \Lambda^\ell) \longrightarrow L^p_\alpha(\Omega, \Lambda^\ell), \tag{2.8}$$

and if

$$\overset{\circ}{L}{}^p_s(\Omega, \Lambda^\ell) := \text{ the closure of } C^\infty_o(\Omega, \Lambda^\ell) \text{ in } L^p_s(\Omega, \Lambda^\ell), \quad p \in (1, \infty), \ s \in \mathbb{R}, \tag{2.9}$$

then

$$\mathrm{Ker}\,(\mathrm{Tr}) = \overset{\circ}{L}{}^p_s(\Omega, \Lambda^\ell), \quad 1 < p < \infty, \ \tfrac{1}{p} < \alpha < 1 + \tfrac{1}{p}. \tag{2.10}$$

For $1 < p < \infty$, $s \in \mathbb{R}$, and $\ell \in \{0, 1, ..., n\}$ we next introduce

$$\mathcal{D}^p_\ell(\Omega; d) := \{u \in L^p(\Omega, \Lambda^\ell) : \, du \in L^p(\Omega, \Lambda^{\ell+1}\}, \tag{2.11}$$

$$\mathcal{D}^p_\ell(\Omega; \delta) := \{u \in L^p(\Omega, \Lambda^\ell) : \, \delta u \in L^p(\Omega, \Lambda^{\ell-1})\}, \tag{2.12}$$

equipped with the natural graph norms. Throughout the paper, all derivatives are taken in the sense of distributions.

Inspired by (2.3), if $1 < p < \infty$ and $u \in \mathcal{D}^p_\ell(\Omega; \delta)$ we then define $\nu \vee u \in B^{p,p}_{-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1})$ by

$$\langle \nu \vee u, \varphi \rangle := -\langle \delta u, \Psi \rangle + \langle u, d\Psi \rangle \tag{2.13}$$

for any $\varphi \in B^{p',p'}_{\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1})$, $1/p + 1/p' = 1$, and any $\Phi \in L^{p'}_1(\Omega, \Lambda^{\ell-1})$ with $\mathrm{Tr}\,\Phi = \phi$. Note that (2.4), (2.9) imply that he operator

$$\nu \vee \cdot \, : \mathcal{D}^p_\ell(\Omega; \delta) \longrightarrow B^{p,p}_{-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1}) \tag{2.14}$$

is well-defined, linear and bounded for each $p \in (1, \infty)$. i.e.

$$\|\nu \vee u\|_{B^{p,p}_{-\frac{1}{p}}(\partial\Omega, \Lambda^{\ell-1})} \leq C\Big(\|u\|_{L^p_s(\Omega, \Lambda^\ell)} + \|\delta u\|_{L^p_s(\Omega, \Lambda^{\ell-1})}\Big). \tag{2.15}$$

The range of the operator (2.14) will be denoted by

$$\mathcal{X}^p_\ell(\partial\Omega) := \Big\{\nu \vee u : \, u \in \mathcal{D}^p_{\ell+1}(\Omega; \delta)\Big\} \hookrightarrow B^{p,p}_{-\frac{1}{p}}(\partial\Omega, \Lambda^\ell), \tag{2.16}$$

which we equip with natural "infimum" norm. It follows that the operator

$$\begin{aligned}&\delta_\partial : \mathcal{X}^p_\ell(\partial\Omega) \longrightarrow \mathcal{X}^p_{\ell-1}(\partial\Omega) \\ &\delta_\partial f := -\nu \vee \delta w, \quad \text{if} \quad f = \nu \vee w, \quad w \in \mathcal{D}^p_{\ell+1}(\Omega;\delta)\end{aligned} \tag{2.17}$$

is well-defined, linear and bounded.

Other spaces of interest for us here are defined as follows. For $1 < p < \infty$, $s \in \mathbb{R}$, and $\ell \in \{0, 1, ..., n\}$, consider

$$\mathcal{D}^p_\ell(\Omega;\delta_\vee) := \{u \in L^p(\Omega,\Lambda^\ell) : \ \delta u \in L^p(\Omega,\Lambda^{\ell-1}), \ \nu \vee u = 0\}, \tag{2.18}$$

once again equipped with the natural graph norm.

For further use, we record here a useful variation on (2.3), namely that if $1 < p, p' < \infty$ satisfy $1/p + 1/p' = 1$ then

$$\langle du, v\rangle = \langle u, \delta v\rangle, \quad \forall\, u \in \mathcal{D}^p_\ell(\Omega; d), \ \forall\, v \in \mathcal{D}^{p'}_\ell(\Omega;\delta_\vee). \tag{2.19}$$

2.3 The $\{\mathcal{V}, \mathcal{H}, a\}$ formalism

Let $\mathcal{V}$ be a reflexive Banach space continuously and densely embedded into a Hilbert space $\mathcal{H}$ so that, in particular,

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^* \tag{2.20}$$

and assume that

$$a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{C} \tag{2.21}$$

is a sesqui-linear, bounded form. Then

$$A_o : \mathcal{V} \longrightarrow \mathcal{V}^*, \qquad A_o u := a(u,\cdot) \in \mathcal{V}^*, \quad \forall\, u \in \mathcal{V}, \tag{2.22}$$

is a linear, bounded operator satisfying

$${}_{\mathcal{V}^*}\langle A_o u, v\rangle_{\mathcal{V}} = a(u, v), \quad \forall\, u, v \in \mathcal{V}. \tag{2.23}$$

Assume further that $a(\cdot,\cdot)$ is symmetric and coercive, in the sense that there exist $C_1, C_2 > 0$ such that

$$\mathrm{Re}\, a(u,u) + C_1\|u\|^2_{\mathcal{H}} \geq C_2\|u\|^2_{\mathcal{V}}, \qquad \forall\, u \in \mathcal{V}. \tag{2.24}$$

Then

$$A_o : \mathcal{V} \longrightarrow \mathcal{V}^* \quad \text{is bounded and self-adjoint.} \tag{2.25}$$

Furthermore, A_o is invertible if the constant C_1 appearing in (2.24) can be taken to be zero.

Going further, take A to be the *part of* A_o *in* $\mathcal{H}$, i.e.

$$A := A_o\Big|_{\mathrm{Dom}\,(A)} : \mathcal{H} \longrightarrow \mathcal{H} \tag{2.26}$$

where

$$\mathrm{Dom}\,(A) := \{u \in \mathcal{V} : \ A_o u \in \mathcal{H}\}. \tag{2.27}$$

Hence, (2.26)-(2.27) is an unbounded, self-adjoint operator on $\mathcal{H}$. Furthermore, there exists $\theta \in (0, \pi)$ such that

$$\|(\lambda I - A)^{-1}\| \leq \frac{C}{|\lambda|}, \qquad |\arg(\lambda)| < \pi - \theta, \tag{2.28}$$

i.e., A is *sectorial.* In particular, according to the classical Hille-Yoshida theorem, the operator A generates an analytic semigroup on $\mathcal{H}$ (see, e.g., [14]). Finally, when A is invertible (which is the case if we can take $C_1 = 0$ in (2.24)), then the semigroup $\left(e^{tA}\right)_{t>0}$ is bounded.

3 The Hodge-Laplacian and related operators

3.1 The Hodge-Laplacian

Recall that the ℓ-th *Betti number* of Ω, denoted by $b_\ell(\Omega)$, is defined as the dimension the ℓ-th singular homology group of Ω, viewed as a topological space, over the reals. It has been proved if [11] that for each Lipschitz domain $\Omega \subset \mathcal{M}$ there exist two conjugate exponents

$$1 \leq p_\Omega < 2 < q_\Omega \leq \infty \tag{3.1}$$

such that the space

$$\mathcal{H}^p(\Omega, \Lambda^\ell) := \{u \in L^p(\Omega, \Lambda^\ell) : \ du = 0, \ \delta u = 0, \ \nu \vee u = 0\} \tag{3.2}$$

is independent of p if $p \in (p_\Omega, q_\Omega)$ and has dimension $b_\ell(\Omega)$. We shall occasionally abbreviate $\mathcal{H}(\Omega, \Lambda^\ell) := \mathcal{H}^2(\Omega, \Lambda^\ell)$.

Consequently, the orthogonal projection of $L^2(\Omega, \Lambda^\ell)$ onto $\mathcal{H}(\Omega, \Lambda^\ell)$ extends canonically to a bounded operator

$$P_p : L^p(\Omega, \Lambda^\ell) \longrightarrow \mathcal{H}(\Omega, \Lambda^\ell) \hookrightarrow L^p(\Omega, \Lambda^\ell) \text{ if } p_\Omega < p < q_\Omega, \tag{3.3}$$

which has the property that

$$P_p^* = P_{p'}, \qquad p, p' \in (p_\Omega, q_\Omega), \;\; 1/p + 1/p' = 1. \tag{3.4}$$

In order to continue, for each $p \in (1, \infty)$ and $\ell \in \{0, 1, ..., n\}$ we set

$$\begin{aligned} \mathcal{V}^p(\Omega, \Lambda^\ell) \;&:=\; \{u \in L^p(\Omega, \Lambda^\ell) : \, du \in L^p(\Omega, \Lambda^{\ell+1}), \; \delta u \in L^p(\Omega, \Lambda^{\ell-1}), \; \nu \vee u = 0\} \\ &=\; \mathcal{D}^p(\Omega; d) \cap \mathcal{D}^p(\Omega; \delta_\vee) \end{aligned} \tag{3.5}$$

once again equipped with the natural graph norm. If $p = 2$, we introduce the quadratic form

$$Q_\ell(u, v) := \langle du, dv \rangle + \langle \delta u, \delta v \rangle, \quad u, v \in \mathcal{V}^2(\Omega, \Lambda^\ell). \tag{3.6}$$

Proposition 3.1. *If $b_\ell(\Omega) = 0$ then the quadratic form Q_ℓ is coercive on $\mathcal{V}^2(\Omega, \Lambda^\ell)$, in the sense that there exists $C > 0$ such that*

$$Q_\ell(u, u) \geq C \|u\|^2_{L^2(\Omega, \Lambda^\ell)}, \qquad \forall \, u \in \mathcal{V}^2(\Omega, \Lambda^\ell). \tag{3.7}$$

Proof. We shall rely on the estimate

$$\|u\|^2_{L^2_{1/2}(\Omega, \Lambda^\ell)} \leq C \Big(\|u\|^2_{L^2(\Omega, \Lambda^\ell)} + Q_\ell(u, u) \Big), \quad \forall \, u \in \mathcal{V}^2(\Omega, \Lambda^\ell), \tag{3.8}$$

which has been established in [11]. Reasoning by contradiction, assume that (3.7) fails, so that there exists a sequence $u_j \in \mathcal{V}^2(\Omega, \Lambda^\ell)$ such that $\|u_j\|_{L^2(\Omega, \Lambda^\ell)} = 1$ for each j and for which $Q_\ell(u_j, u_j) \to 0$ as $j \to \infty$. By (3.8) and Rellich's selection lemma, there is no loss of generality in assuming that

$$u_j \to u \text{ in } L^2(\Omega, \Lambda^\ell) \text{ for some } u \text{ with } \|u\|_{L^2(\Omega, \Lambda^\ell)} = 1. \tag{3.9}$$

Now, $Q_\ell(u_j, u_j) \to 0$ as $j \to \infty$ forces $du_j \to 0$ and $\delta u_j \to 0$ in $L^2(\Omega, \Lambda^{\ell+1})$ and $L^2(\Omega, \Lambda^{\ell-1})$, respectively, as $j \to \infty$ and, hence, $du = 0$, $\delta u = 0$. Furthermore, by the continuity of the operator (2.14), we also have $\nu \vee u = 0$. Consequently, $u \in \mathcal{H}(\Omega, \Lambda^\ell) = \{0\}$, given that $b_\ell(\Omega) = 0$. This contradicts (3.9) and proves the proposition. $\square$

Thus, under the assumption that $b_\ell(\Omega) = 0$, the space $\mathcal{V}^2(\Omega, \Lambda^\ell)$ has a Hilbert structure when equipped with the inner product $Q_\ell(\cdot, \cdot)$. The Friedrichs extension method discussed in §2.3 for $\mathcal{V} := \mathcal{V}^2(\Omega, \Lambda^\ell)$, $\mathcal{H} := L^2(\Omega, \Lambda^\ell)$ and $a(u, v) := Q_\ell(u, v)$, then yields an unbounded self-adjoint operator

$$B : L^2(\Omega, \Lambda^\ell) \longrightarrow L^2(\Omega, \Lambda^\ell) \tag{3.10}$$

whose domain, Dom$\,(B)$, consists of all

$$\begin{aligned}&u \in \mathcal{V}^2(\Omega,\Lambda^\ell) \text{ such that there exists } C>0 \text{ with the property that}\\ &\qquad |Q_\ell(u,v)| \leq C\|v\|_{L^2(\Omega,\Lambda^\ell)} \text{ for all } v \in \mathcal{V}^2(\Omega,\Lambda^\ell)\end{aligned} \tag{3.11}$$

and for which

$$\langle Bu, v\rangle = Q_\ell(u,v), \qquad u \in \mathrm{Dom}\,(B),\ v \in \mathcal{V}^2(\Omega,\Lambda^\ell). \tag{3.12}$$

If we now regard $d : L^2(\Omega,\Lambda^\ell) \longrightarrow L^2(\Omega,\Lambda^{\ell+1})$ as an unbounded operator with domain $\mathcal{D}^2_\ell(\Omega;d)$, it is not difficult to check that

$$B = dd^* + d^*d \tag{3.13}$$

in the sense of composition of unbounded operators.

The latest description of B has a natural analogue in the L^p context, as follows. If $1<p<\infty$, define the unbounded operator

$$B_p : L^p(\Omega,\Lambda^\ell) \longrightarrow L^p(\Omega,\Lambda^\ell) \tag{3.14}$$

with domain $\mathrm{Dom}\,(B_p)$ consisting of

$$u \in \mathcal{D}^p_\ell(\Omega;d) \cap \mathcal{D}^p_\ell(\Omega;\delta_\vee) \text{ with } du \in \mathcal{D}^p_{\ell+1}(\Omega;\delta_\vee),\ \delta u \in \mathcal{D}^p_{\ell+1}(\Omega;d) \tag{3.15}$$

by setting

$$B_p u := -\Delta u = (d\delta + \delta d)u, \qquad \forall\, u \in \mathrm{Dom}\,(B_p). \tag{3.16}$$

Note that since $C^\infty_o(\Omega,\Lambda^\ell)$ is contained in $\mathrm{Dom}\,(B_p)$, it follows that B_p is densely defined.

Proposition 3.2. *Let $\Omega \subset \mathcal{M}$ be a Lipschitz domain. Then for each $p_\Omega < p < q_\Omega$ there exists a linear, bounded operator*

$$G_p : L^p(\Omega,\Lambda^\ell) \longrightarrow L^p(\Omega,\Lambda^\ell), \tag{3.17}$$

such that $\mathrm{Im}\,(G_p) \subset \mathrm{Dom}\,(B_p)$,

$$\begin{aligned}&\|G_p u\|_{L^p(\Omega,\Lambda^\ell)} + \|dG_p u\|_{L^p(\Omega,\Lambda^{\ell+1})} + \|\delta G_p u\|_{L^p(\Omega,\Lambda^{\ell-1})}\\ &\qquad + \|d\delta G_p u\|_{L^p(\Omega,\Lambda^\ell)} + \|\delta d G_p u\|_{L^p(\Omega,\Lambda^\ell)} \leq C\|u\|_{L^p(\Omega,\Lambda^\ell)}\end{aligned} \tag{3.18}$$

and, in the sense of composition of unbounded operators,

$$B_pG_p = G_pB_p = I - P_p \text{ on } L^p(\Omega, \Lambda^\ell). \tag{3.19}$$

Furthermore,

$$(G_p)^* = G_{p'}, \quad p_\Omega < p, p' < q_\Omega, \; 1/p + 1/p' = 1 \tag{3.20}$$

and (with the subscript ℓ used to indicate the dependence on the degree),

$$dG_{p,\ell} = G_{p,\ell+1}d \text{ on } \mathcal{D}^p_\ell(\Omega; d), \tag{3.21}$$

$$\delta G_{p,\ell} = G_{p,\ell-1}\delta \text{ on } \mathcal{D}^p_\ell(\Omega; \delta). \tag{3.22}$$

Such a Green operator has been constructed in [13] when $n = 3$ and in [12] in the general case. Let us also note here that (3.19) implies

$$\mathrm{Ker}\,(B_p) = \mathcal{H}^p(\Omega, \Lambda^\ell) \text{ whenever } p_\Omega < p < q_\Omega, \tag{3.23}$$

and that, thanks to (2.19),

$$\mathrm{Im}\,(B_p) \hookrightarrow \mathrm{Ker}\,(P_p) \text{ for each } p \in (p_\Omega, q_\Omega). \tag{3.24}$$

Proposition 3.3. *For each Lipschitz subdomain Ω of $\mathcal{M}$ there holds*

$$(B_p)^* = B_{p'}, \quad p_\Omega < p, p' < q_\Omega, \; 1/p + 1/p' = 1. \tag{3.25}$$

Proof. The inclusion $B_{p'} \subset (B_p)^*$ is immediate from definitions, so it remains to prove the opposite one. To this end, if $u \in \mathrm{Dom}\,(B_p^*)$, then $u \in L^p(\Omega, \Lambda^\ell)$ and there exists $w \in L^{p'}(\Omega, \Lambda^\ell)$ such that

$$\langle w, v\rangle = \langle u, B_p v\rangle, \qquad \forall\, v \in \mathrm{Dom}\,(B_p). \tag{3.26}$$

Choosing $v := P_p\xi$ with $\xi \in L^p(\Omega, \Lambda^\ell)$ arbitrary forces $\langle w, P_p\xi\rangle = 0$ and, ultimately, $P_{p'}w = 0$. Next, pick $v := G_p\eta$ with $\eta \in L^p(\Omega, \Lambda^\ell)$ arbitrary and write

$$\begin{aligned} \langle G_{p'}w, \eta\rangle &= \langle w, G_{p'}\eta\rangle = \langle w, v\rangle \\ &= \langle u, B_p v\rangle = \langle u, B_pG_p\eta\rangle = \langle u, (I - P_p)\eta\rangle \\ &= \langle (I - P_p)u, \eta\rangle. \end{aligned} \tag{3.27}$$

Since η is arbitrary, this forces $(I - P_p)u = G_{p'}w$ and, further, $u = G_{p'}w + P_{p'}u \in \mathrm{Dom}\,(B_{p'})$. In addition, $B_{p'}u = B_{p'}G_{p'}w + B_{p'}P_{p'}u = (I - P_{p'})w = w$ which shows that $(B_p)^* \subset B_{p'}$. This finishes the proof of the proposition. $\square$

Going further, we now define the Leray projection operator

$$\mathbb{P}_p := \delta d G_p + P_p, \qquad p_\Omega < p < q_\Omega, \tag{3.28}$$

and introduce the spaces

$$X^p(\Omega, \Lambda^\ell) \;:=\; \{u \in \mathcal{D}^p_\ell(\Omega; \delta_\vee) : \delta u = 0\}, \tag{3.29}$$

$$Y^p(\Omega, \Lambda^\ell) \;:=\; \{du : u \in \mathcal{D}^p_{\ell-1}(\Omega; d)\}. \tag{3.30}$$

As a result of the Hodge decompositions proved in [12], there holds

$$L^p(\Omega, \Lambda^\ell) = X^p(\Omega, \Lambda^\ell) \oplus Y^p(\Omega, \Lambda^\ell), \qquad p_\Omega < p < q_\Omega. \tag{3.31}$$

For further reference we also note the following.

Lemma 3.4. *For each $p_\Omega < p, p' < q_\Omega$ with $1/p + 1/p' = 1$, the natural integral pairing on Ω allows for the identification*

$$\Big(X^p(\Omega, \Lambda^\ell)\Big)^* \equiv X^{p'}(\Omega, \Lambda^\ell). \tag{3.32}$$

Proof. The goal is to show that the mapping

$$\Phi : X^{p'}(\Omega, \Lambda^\ell) \ni u \mapsto \langle u, \cdot\rangle \in \Big(X^p(\Omega, \Lambda^\ell)\Big)^* \tag{3.33}$$

is an isomorphism. To see that it is one-to-one, assume that $w \in L^p(\Omega, \Lambda^\ell)$ is arbitrary and, using (3.31), decompose $w = w_1 + w_2$ with $w_1 \in X^p(\Omega, \Lambda^\ell)$ and $w_2 \in Y^p(\Omega, \Lambda^\ell)$. Then, if $u \in X^{p'}(\Omega, \Lambda^\ell)$ is such that $\Phi(u) = 0$, it follows that $\langle u, w\rangle = \langle u, w_1\rangle + \langle u, w_2\rangle = 0$, by (2.19) and (3.29)-(3.30). Since w was arbitrary, this forces $u = 0$ and, hence, Φ is one-to-one.

To prove that the mapping (3.33) is onto, let $f \in \Big(X^p(\Omega, \Lambda^\ell)\Big)^*$ be arbitrary. Since $X^p(\Omega, \Lambda^\ell)$ is a closed subspace of $L^p(\Omega, \Lambda^\ell)$, Hahn-Banach extension theorem in concert with Riesz's representation theorem imply that there exists $w \in L^{p'}(\Omega, \Lambda^\ell)$ such that $f(u) = \langle u, w\rangle$ for each $u \in X^p(\Omega, \Lambda^\ell)$. Invoking (3.31), we once again decompose $w = w_1 + w_2$ with $w_1 \in X^{p'}(\Omega, \Lambda^\ell)$ and $w_2 \in Y^{p'}(\Omega, \Lambda^\ell)$. Since, as before, $\langle u, w_2\rangle = 0$ whenever $u \in X^p(\Omega, \Lambda^\ell)$, we may conclude that $\Phi(w_1) = f$. This proves that Φ is also onto, hence an isomorphism. □

Proposition 3.5. *For each $p \in (p_\Omega, q_\Omega)$, the operator $\mathbb{P}_p$ introduced in (3.28) maps*

$$\mathbb{P}_p : L^p(\Omega, \Lambda^\ell) \longrightarrow X^p(\Omega, \Lambda^\ell) \tag{3.34}$$

in a bounded, linear fashion, and satisfies

$$\mathbb{P}_p^2 = \mathbb{P}_p, \qquad (\mathbb{P}_p)^* = \mathbb{P}_{p'} \tag{3.35}$$

if $1/p + 1/p' = 1$.

Proof. The first part in the statement of the proposition follows from the fact that G_p maps $L^p(\Omega, \Lambda^\ell)$ into $\mathrm{Dom}\,(B_p)$ .As for (3.34), we first note that thanks to (3.19),

$$I - \mathbb{P}_p = d\delta G_p. \tag{3.36}$$

Based on this, if $u \in L^p(\Omega, \Lambda^\ell)$ and $v \in L^{p'}(\Omega, \Lambda^\ell)$ with $p_\Omega < p, p' < q_\Omega$, we obtain

$$\langle \mathbb{P}_p u, (I - \mathbb{P}_p)v \rangle = \langle \mathbb{P}_p, d\delta G_{p'} v \rangle = 0 \tag{3.37}$$

where in the last step we have used (2.19) and (3.29). Thus, $\langle \mathbb{P}_p u, v \rangle = \langle \mathbb{P}_p u, \mathbb{P}_{p'} v \rangle$ and, since the last expression is symmetric in u and v, it follows that $\langle \mathbb{P}_p u, v \rangle = \langle u, \mathbb{P}_{p'} v \rangle$. Hence, $(\mathbb{P}_p)^* = \mathbb{P}_{p'}$. Armed with this, we may now write $\langle \mathbb{P}_p^2 u, v \rangle = \langle \mathbb{P}_p u, \mathbb{P}_{p'} v \rangle = \langle \mathbb{P}_p u, v \rangle$ so that $\mathbb{P}_p^2 = \mathbb{P}_p$, as desired. $\square$

Lemma 3.6. *For each* $p_\Omega < p < q_\Omega$,

$$\mathrm{Dom}\,(B_p) \cap X^p(\Omega, \Lambda^\ell) = \{u \in \mathrm{Dom}\,(B_p) :\ B_p u \in X^p(\Omega, \Lambda^\ell)\}. \tag{3.38}$$

Proof. Let $u \in \mathrm{Dom}\,(B_p) \cap X^p(\Omega, \Lambda^\ell)$, be arbitrary. Then $B_p u = \delta d u$ satisfies $\delta(\delta d u) = 0$ and $\nu \vee \delta d u = -\delta_\partial(\nu \vee du) = 0$. Thus $B_p u \in X^p(\Omega, \Lambda^\ell)$, proving the left-to-right inclusion in (3.38). To prove the opposite one, assume that $u \in \mathrm{Dom}\,(B_p)$ has the property that $B_p u \in X^p(\Omega, \Lambda^\ell)$. Then $0 = \nu \vee (\delta d + d\delta)u = \nu \vee d\delta u$ and $0 = \delta(\delta d + d\delta)u = \delta(d\delta u) = \Delta(\delta u)$. Since $\nu \vee \delta u = -\delta_\partial(\nu \vee u) = 0$, it follows that $\delta u \in \mathrm{Ker}\,(B_p) = \mathcal{H}^p_\vee(\Omega, \Lambda^\ell)$, by (3.23). From this we may deduce that $w := \delta u$ satisfies $dw = 0$ and that $w \in L^q(\Omega, \Lambda^{\ell-1})$ for each $p_\Omega < q < q_\Omega$. Membership of u to $\mathrm{Dom}\,(B_p)$ also guarantees that $\nu \vee u = 0$. In particular, the integration by parts formula (2.19) applies and gives that $\langle \delta u, \delta u \rangle = \langle \delta u, w \rangle = 0$. In turn, this forces $\delta u = 0$ which, further, entails $u \in X^p(\Omega, \Lambda^\ell)$. $\square$

Lemma 3.7. *For each* $p_\Omega < p < q_\Omega$,

$$\mathbb{P}_p B_p = B_p \mathbb{P}_p \quad \textit{on} \quad \mathrm{Dom}\,(B_p). \tag{3.39}$$

Proof. We first claim that if $u \in \mathrm{Dom}\,(B_p)$ then $d\delta G_p u \in \mathrm{Dom}\,(B_p)$. Indeed,

$$
\begin{aligned}
& d\delta G_p u \in L^p(\Omega, \Lambda^\ell), \\
& d(d\delta G_p u) = 0 \in L^p(\Omega, \Lambda^{\ell+1}), \\
& \delta(d\delta G_p u) = -\delta\Delta G_p u = -\delta(I - P_p)u = -\delta u \in L^p(\Omega, \Lambda^{\ell-1}), \\
& (d\delta)(d\delta G_p u) = d[\delta(d\delta G_p u)] = -d\delta u \in L^p(\Omega, \Lambda^\ell), \\
& (\delta d)(d\delta G_p u) = 0 \in L^p(\Omega, \Lambda^\ell),
\end{aligned} \tag{3.40}
$$

and $\nu \vee d(d\delta G_p u) = 0$,

$$
\nu \vee d\delta G_p u = -\nu\delta dG_p u - \nu \vee u + \nu \vee P_p u = 0 - \delta_\partial(\nu \vee dG_p u) - 0 = 0, \tag{3.41}
$$

justifying the claim. Consequently,

$$
\mathbb{P}_p u = u - \delta dG_p u \in \mathrm{Dom}\,(B_p) \text{ if } u \in \mathrm{Dom}\,(B_p), \tag{3.42}
$$

or, in other words,

$$
\mathbb{P}_p : \mathrm{Dom}\,(B_p) \longrightarrow \mathrm{Dom}\,(B_p) \cap X^p(\Omega, \Lambda^\ell). \tag{3.43}
$$

Furthermore, for every $u \in \mathrm{Dom}\,(B_p)$, we may write

$$
\begin{aligned}
B_p \mathbb{P}_p u &= -\Delta(u - d\delta G_p u) \\
&= d\delta u + \delta du + d\delta\Delta G_p u = d\delta u + \delta du + d\delta(P_p - I)u \\
&= \delta du.
\end{aligned} \tag{3.44}
$$

On the other hand, for every $u \in \mathrm{Dom}\,(B_p)$,

$$
\mathbb{P}_p B_p u = \delta dG_p B_p u + P_p B_p u = \delta d(u - P_p u) + 0 = \delta du \tag{3.45}
$$

which, in concert with (3.44), proves (3.39). $\square$

Lemma 3.8. *If $p_\Omega < p < q_\Omega$, then for each $0 \le \ell \le n$,*

$$
d = d\,\mathbb{P}_{p,\ell} + \mathbb{P}_{p,\ell+1}\,d \quad \text{on} \quad \mathcal{D}^p_\ell(\Omega; d), \tag{3.46}
$$

where $\mathbb{P}_{p,\ell}$ stands for $\mathbb{P}_p$ acting on $L^p(\Omega, \Lambda^\ell)$, etc.

Proof. Since $P_{p,\ell+1}(du) = 0$ and $dP_{p,\ell}u = 0$ for every $u \in \mathcal{D}^p_\ell(\Omega; d)$, based on (3.28), (3.22) and (3.19) we may write

$$\begin{aligned} d\,\mathbb{P}_{p,\ell}\,u &= d\delta G_{p,\ell+1}(du) = -\delta d G_{p,\ell+1}(du) - \Delta G_{p,\ell+1}(du) \\ &= -\mathbb{P}_{p,\ell+1}(du) + du, \qquad \forall\, u \in \mathcal{D}^p_\ell(\Omega; d), \end{aligned} \tag{3.47}$$

from which (3.46) follows. □

3.2 The Stokes operator

Given a Lipschitz subdomain Ω of $\mathcal{M}$ and $1 < p < \infty$, let us now define the *Stokes operator*

$$A_p : X^p(\Omega, \Lambda^\ell) \longrightarrow X^p(\Omega, \Lambda^\ell) \tag{3.48}$$

by setting

$$\begin{aligned} &\mathrm{Dom}\,(A_p) := \mathrm{Dom}\,(B_p) \cap X^p(\Omega, \Lambda^\ell) \\ &A_p u := B_p u = -\Delta u, \quad \forall\, u \in \mathrm{Dom}\,(A_p). \end{aligned} \tag{3.49}$$

Lemma 3.9. *For each $p_\Omega < p < q_\Omega$,*

$$A_p \mathbb{P}_p = \mathbb{P}_p B_p \quad \text{on} \quad \mathrm{Dom}\,(B_p). \tag{3.50}$$

Proof. By relying on Lemma 3.7 and the fact that $A_p = B_p$ on $\mathrm{Dom}\,(B_p) \cap X^p(\Omega, \Lambda^\ell)$, on $\mathrm{Dom}\,(B_p)$ we may write

$$A_p \mathbb{P}_p = B_p \mathbb{P}_p = \mathbb{P}_p B_p \tag{3.51}$$

where (3.43) is also used. □

For a (possibly unbounded) operator T we let $\mathrm{Spec}\,(T)$ denote its spectrum.

Lemma 3.10. *For each $p_\Omega < p < q_\Omega$,*

$$\mathrm{Spec}\,(A_p) \subseteq \mathrm{Spec}\,(B_p) \tag{3.52}$$

and, for each $\lambda \notin \mathrm{Spec}\,(B_p)$,

$$(\lambda I - B_p)^{-1} \mathbb{P}_p = \mathbb{P}_p (\lambda I - B_p)^{-1} = (\lambda I - A_p)^{-1} \mathbb{P}_p \tag{3.53}$$

on $L^p(\Omega, \Lambda^\ell)$.

Proof. If $\lambda \notin \mathrm{Spec}\,(B_p)$ then $(\lambda I - B_p)^{-1}$ is invertible on $L^p(\Omega, \Lambda^\ell)$ and, hence, $\lambda I - A_p$ is one to one. Next, if $f \in X^p(\Omega, \Lambda^\ell) \hookrightarrow L^p(\Omega, \Lambda^\ell)$ is arbitrary, set $w := (\lambda I - B_p)^{-1} f \in \mathrm{Dom}\,(B_p) \hookrightarrow L^p(\Omega, \Lambda^\ell)$. It follows that $u := \mathbb{P}_p w \in \mathrm{Dom}\,(B_p) \cap X^p(\Omega, \Lambda^\ell) = \mathrm{Dom}\,(A_p)$ and

$$(\lambda I - A_p)^{-1} u = \lambda\, \mathbb{P}_p w - A_p \mathbb{P}_p w = \mathbb{P}_p(\lambda I - B_p) w = \mathbb{P}_p f = f \tag{3.54}$$

which proves that $\lambda I - A_p$ is onto as well. Hence, (3.52) holds. Then the commutation identities in (3.53) are straightforward consequences of this and Lemma 3.8. □

Lemma 3.11. *Whenever $p_\Omega < p, p' < q_\Omega$ are such that $1/p + 1/p' = 1$, there holds*

$$(A_p)^* = A_{p'}. \tag{3.55}$$

Proof. With p, p' as in the statement of the lemma, let $u \in \mathrm{Dom}\,(A_p^*) \subset X^{p'}(\Omega, \Lambda^\ell)$ and set $w := A_p^* u \in X^{p'}(\Omega, \Lambda^\ell)$. In particular,

$$\langle w, \eta \rangle = \langle u, A_p \eta \rangle, \quad \forall\, \eta \in \mathrm{Dom}\,(A_p) \subset X^p(\Omega, \Lambda^\ell). \tag{3.56}$$

Then for every $\xi \in \mathrm{Dom}\,(B_p)$ we may write

$$\begin{aligned} \langle B_p \xi, u \rangle &= \langle B_p \xi, \mathbb{P}_{p'} u \rangle = \langle \mathbb{P}_p B_p \xi, u \rangle \\ &= \langle A_p(\mathbb{P}_p \xi), u \rangle = \langle \mathbb{P}_p \xi, w \rangle = \langle \xi, \mathbb{P}_{p'} w \rangle \\ &= \langle \xi, w \rangle \end{aligned} \tag{3.57}$$

where we have used (3.56) and the fact that $\mathbb{P}_p \xi \in \mathrm{Dom}\,(A_p)$ whenever $\xi \in \mathrm{Dom}\,(B_p)$. It then follows from (3.57) that $u \in \mathrm{Dom}\,(B_p^*) = \mathrm{Dom}\,(B_{p'})$ and $B_{p'} u = w \in X^{p'}(\Omega, \Lambda^\ell)$. Thus, $u \in \mathrm{Dom}\,(B_{p'}) \cap X^{p'}(\Omega, \Lambda^\ell) = \mathrm{Dom}\,(A_{p'})$ and $A_{p'} u = B_{p'} u = w = A_p^* u$. In particular, this shows that $(A_p)^* \subseteq A_{p'}$.

Conversely, fix an arbitrary $u \in \mathrm{Dom}\,(A_{p'})$. Hence, $u \in X^{p'}(\Omega, \Lambda^\ell) \cap \mathrm{Dom}\,(B_{p'})$ which then gives

$$\langle A_{p'} u, w \rangle = \langle B_{p'} u, w \rangle = \langle u, B_p w \rangle = \langle u, A_p w \rangle \tag{3.58}$$

for each $w \in X^p(\Omega, \Lambda^\ell) \cap \mathrm{Dom}\,(B_p) = \mathrm{Dom}\,(A_p)$. In turn, this shows that $u \in \mathrm{Dom}\,(A_p^*)$ and $A_p^* = A_{p'} u$ and, further, that $A_{p'} \subseteq A_p^*$. The proof of the lemma is therefore finished. □

Remark. For each $p_\Omega < p < q_\Omega$,

$$\mathrm{Ker}\, A_p = \mathrm{Ker}\, B_p = \mathcal{H}^p(\Omega, \Lambda^\ell) \tag{3.59}$$

so both operators are invertible if $b_\ell(\Omega) = 0$.

Proposition 3.12. *Let $\ell \in \{0,1,...,n\}$ be fixed and consider the spaces*

$$\mathcal{H} := X^2(\Omega,\Lambda^\ell), \quad \mathcal{V} := \{u \in \mathcal{H} :\ du \in L^2(\Omega,\Lambda^{\ell+1})\} \tag{3.60}$$

where the latter space is equipped with the natural graph norm. Also, consider the sesqui-linear form

$$a(u,v) := \langle du, dv\rangle, \qquad u,v \in \mathcal{V}. \tag{3.61}$$

Then the operator associated with the triplet $\{\mathcal{V},\, \mathcal{H},\, a\}$ as in §2.3 is precisely A_2 (i.e., the Stokes operator introduced in (3.48)-(3.49) for $p=2$). In particular, A_2 generates an analytic semigroup on $X^2(\Omega,\Lambda^\ell)$.

Proof. According to the discussion in §2.3, the domain of the operator A associated with the triplet $\{\mathcal{V},\, \mathcal{H},\, a\}$ consists of forms $u \in \mathcal{V}$ for which there exists $w \in \mathcal{H}$ such that

$$\langle du, dv\rangle = \langle w, v\rangle, \qquad \forall\, v \in \mathcal{V}. \tag{3.62}$$

In order to continue, we shall need the fact that the operator

$$\mathbb{P}_2 : \mathcal{D}^2_\ell(\Omega; d) \longrightarrow \mathcal{V} \quad \text{is onto.} \tag{3.63}$$

Indeed, the fact that $\mathbb{P}_2$ maps the space $\mathcal{D}^2_\ell(\Omega; d)$ into $\mathcal{V}$ is clear from Lemma 3.8. Since

$$\mathbb{P}_{2,\ell} \text{ acts as the identity on } \mathcal{H}, \tag{3.64}$$

any $v \in \mathcal{V} = \mathcal{H} \cap \mathcal{D}^2_\ell(\Omega; d)$ can be written as $v = \mathbb{P}_2 v$, proving (3.63).

Returning now to the mainstream discussion, the above analysis shows that if u, w are as in the opening paragraph, then condition (3.62) is equivalent to

$$\langle du, d\mathbb{P}_2 f\rangle = \langle w, \mathbb{P}_2 f\rangle, \qquad \forall\, f \in \mathcal{D}^2_\ell(\Omega; d). \tag{3.65}$$

Now, if $f \in \mathcal{D}^2_\ell(\Omega; d)$ is arbitrary, based on this, (3.64) and (3.47) we may write

$$\begin{aligned}
\langle w, f\rangle &= \langle \mathbb{P}_2 w, f\rangle = \langle w, \mathbb{P}_2 f\rangle = \langle du, d\mathbb{P}_2 f\rangle \\
&= \langle du, -\mathbb{P}_{2,\ell+1}(df) + df\rangle = \langle du, -\mathbb{P}_{2,\ell+1}(df) + \langle du, df\rangle \\
&= \langle -\mathbb{P}_{2,\ell+1}(du), df\rangle + \langle du, df\rangle = \langle -\mathbb{P}_{2,\ell+1}(du) + du, df\rangle \\
&= \langle d\mathbb{P}_{2,\ell}\, u\,,\, df\rangle = \langle du, df\rangle.
\end{aligned} \tag{3.66}$$

In turn, this is equivalent to demanding that $du \in \mathcal{D}^2_{\ell+1}(\Omega; \delta_\vee)$ and $\delta du = w$.

Consequently, the domain of A is precisely the collection of forms

$$u \in L^2(\Omega, \Lambda^\ell), \quad \delta u = 0, \quad \nu \vee u = 0, \quad du \in L^2(\Omega, \Lambda^{\ell+1}), \quad \delta du \in L^2(\Omega, \Lambda^\ell), \tag{3.67}$$

and $Au = \delta du = -\Delta u$ for each u as in (3.67). That is, A coincides with A_2 (introduced in (3.48)-(3.49)), as desired. $\square$

3.3 The Maxwell operator

For an arbitrary Lipschitz subdomain Ω of $\mathcal{M}$ and $p_\Omega < p < q_\Omega$, we introduce the spaces

$$\begin{aligned} Z^p(\Omega, \Lambda^\ell) &:= \{u \in L^p(\Omega, \Lambda^\ell) : \, du = 0\}, \\ W^p(\Omega, \Lambda^\ell) &:= \{\delta u : \, u \in \mathcal{D}^p_\ell(\Omega; \delta_\vee)\} \end{aligned} \tag{3.68}$$

and consider the operator

$$\mathbb{Q}_p := d\delta G_p + P_p : L^p(\Omega, \Lambda^\ell) \longrightarrow L^p(\Omega, \Lambda^\ell). \tag{3.69}$$

It's main properties are summarized in the lemma below.

Lemma 3.13. *For each $p_\Omega < p < q_\Omega$, the following hold:*

(i) $(\mathbb{Q}_p)^* = \mathbb{Q}_{p'}$ *if* $1/p + 1/p' = 1$, *and* $(\mathbb{Q}_p)^2 = \mathbb{Q}_p$;

(ii) $\mathbb{Q}_p B_p = B_p \mathbb{Q}_p$ *on* $\mathrm{Dom}\,(B_p)$;

(iii) $\mathbb{Q}_p : L^p(\Omega, \Lambda^\ell) \to Z^p(\Omega, \Lambda^\ell)$ *is onto;*

(iv) $\mathbb{Q}_p : \mathcal{D}^p_\ell(\Omega; \delta_\vee) \to Z^p(\Omega, \Lambda^\ell) \cap \mathcal{D}^p_\ell(\Omega; \delta_\vee)$ *is onto;*

(v) $\delta = \delta \mathbb{Q}_{p,\ell} + \mathbb{Q}_{p,\ell-1}\delta$ *on* $\mathcal{D}^p_\ell(\Omega; \delta_\vee)$.

We omit the proof, which can be carried out much as for the case of the Leray projection $\mathbb{P}_p$.

Next, for each $p_\Omega < p < q_\Omega$, the *Maxwell operator* is introduced as the part of B_p in $Z^p(\Omega, \Lambda^\ell)$, i.e., as the unbounded operator

$$C_p : Z^p(\Omega, \Lambda^\ell) \longrightarrow Z^p(\Omega, \Lambda^\ell) \tag{3.70}$$

for which

$$\begin{aligned} &\mathrm{Dom}\,(C_p) := \mathrm{Dom}\,(B_p) \cap Z^p(\Omega, \Lambda^\ell) \\ &C_p u := B_p u = -\Delta u, \quad \forall\, u \in \mathrm{Dom}\,(C_p). \end{aligned} \tag{3.71}$$

Some of the most basic properties of this operator are summarized below. They parallel those of the Stokes operator discussed in §3.2 and can be proved much in the same fashion.

Lemma 3.14. *For each $p_\Omega < p < q_\Omega$, the following hold.*

(i) $C_p\mathbb{Q}_p = \mathbb{Q}_p B_p$ *on* $\mathrm{Dom}\,(B_p)$*;*

(ii) $\mathrm{Spec}\,(C_p) \subseteq \mathrm{Spec}\,(B_p)$ *and, for each* $\lambda \notin \mathrm{Spec}\,(B_p)$*,*

$$(\lambda I - B_p)^{-1}\mathbb{Q}_p = \mathbb{Q}_p(\lambda I - B_p)^{-1} = (\lambda I - C_p)^{-1}\mathbb{Q}_p \text{ on } L^p(\Omega, \Lambda^\ell); \tag{3.72}$$

(iii) $\Big(Z^p(\Omega, \Lambda^\ell)\Big)^* = Z^{p'}(\Omega, \Lambda^\ell)$ *and* $(C_p)^* = C_{p'}$ *whenever* $1/p + 1/p' = 1$*;*

(iv) $\mathrm{Ker}\, C_p = \mathrm{Ker}\, B_p = \mathcal{H}^p(\Omega, \Lambda^\ell)$ *so that, in particular, the operator (3.70)-(3.71) is invertible if* $b_\ell(\Omega) = 0$*;*

Finally, we remark that, in the case when $p = 2$, the Maxwell operator (3.70)-(3.71) generates an analytic semigroup on $Z^2(\Omega, \Lambda^\ell)$. More specifically, we have the following.

Proposition 3.15. *Fix $\ell \in \{0, 1, ..., n\}$ and consider the spaces*

$$\mathcal{H} := Z^2(\Omega, \Lambda^\ell), \quad \mathcal{V} := \{u \in \mathcal{H} :\, \delta u \in L^2(\Omega, \Lambda^{\ell+1}),\ \nu \vee u = 0\} \tag{3.73}$$

where the latter space is equipped with the natural graph norm. Also, consider the sesqui-linear form

$$a(u, v) := \langle \delta u, \delta v\rangle, \qquad u, v \in \mathcal{V}. \tag{3.74}$$

Then the operator associated with the triplet $\{\mathcal{V},\, \mathcal{H},\, a\}$ as in §2.3 is precisely the Maxwell operator C_2. In particular, C_2 generates an analytic semigroup on $Z^2(\Omega, \Lambda^\ell)$.

The proof of this results can be carried out much as the proof of Proposition 3.12, with the help of Lemma 3.14.

4 The regularity of differential forms in $\mathrm{Dom}\,(B_p)$

We first record a number of useful results from [12], [13].

For each $0 \le \ell \le n$, the operator $-\Delta_\ell = -\Delta : L^2_1(\mathcal{M}, \Lambda^\ell) \to L^2_{-1}(\mathcal{M}, \Lambda^\ell)$ is bounded, nonnegative, and self-adjoint. Since for $\lambda \in \mathbb{R}$ with $\lambda > 0$, the operator $(\lambda I - \Delta_\ell)^{-1}$ is positive, self-adjoint and compact operator on $L^2(\mathcal{M}, \Lambda^\ell)$ it follows that there exists $\mathrm{Spec}\,(\Delta_\ell) \subseteq (-\infty, 0]$, a discrete set such that

$$z \notin \mathrm{Spec}\,(\Delta_\ell) \Rightarrow (\Delta_\ell - zI) : L^2_1(\mathcal{M}, \Lambda^\ell) \longrightarrow L^2_{-1}(\mathcal{M}, \Lambda^\ell) \text{ is invertible.} \tag{4.1}$$

Set

$$\mathcal{U} := \bigcup_{0 \le \ell \le n} \mathrm{Spec}\,(\Delta_\ell) \subset (-\infty, 0], \tag{4.2}$$

and for $\lambda \notin \mathcal{U}$, let $\Gamma_{\lambda,\ell}$ be the Schwartz kernel of $\Delta - \lambda I$ on ℓ-forms. In particular, we denote by $\Pi_{\lambda,\ell}$ the associated (volume) Newtonian potential. Also, once a Lipschitz domain $\Omega \subset \mathcal{M}$ has been fixed, we define the single layer potential operator on $\partial\Omega$ by

$$\mathcal{S}_{\lambda,\ell} f(x) := \int_{\partial\Omega} \langle \Gamma_{\lambda,\ell}(x, y), f(y) \rangle \, d\sigma_y, \qquad x \in \Omega. \tag{4.3}$$

for any $f : \partial\Omega \to \Lambda^\ell$. Note that $(\Delta - \lambda I)\mathcal{S}_{\lambda,\ell} = 0$ in Ω and, as proved in [12], the operators

$$\mathcal{S}_{\lambda,\ell} : \mathcal{X}^p_\ell(\partial\Omega) \longrightarrow L^p_1(\Omega, \Lambda^\ell), \tag{4.4}$$

$$d\mathcal{S}_{\lambda,\ell} : \mathcal{X}^p_\ell(\partial\Omega) \longrightarrow \mathcal{D}^p_{\ell+1}(\Omega; \delta), \tag{4.5}$$

are well-defined and bounded. As a consequence, the operator $M_{\lambda,\ell}$ defined by the equality

$$\left(-\tfrac{1}{2} I + M_{\lambda,\ell}\right) f = \nu \vee (d\,\mathcal{S}_{\lambda,\ell} f), \qquad \forall\, f \in \mathcal{X}^p_\ell(\partial\Omega). \tag{4.6}$$

is well-defined and bounded on $\mathcal{X}^p_\ell(\partial\Omega)$ for each $p \in (1, \infty)$. It has also been shown in [12] that if $0 \le \ell \le n$, $\lambda \notin \mathcal{U}$ and $p_\Omega < p < q_\Omega$,

$$-\tfrac{1}{2} I + M_{\lambda,\ell} : \mathcal{X}^p_\ell(\partial\Omega) \longrightarrow \mathcal{X}^p_\ell(\partial\Omega) \text{ is an isomorphism.} \tag{4.7}$$

Our final remarks is that the spectrum of B_p acting on $L^p(\Omega, \Lambda^\ell)$ is a discrete subset $\mathrm{Spec}\,(B_\ell)$ of $(-\infty, 0]$ which is independent of $p \in (p_\Omega, q_\Omega)$; cf. [12]. We then set

$$\mathcal{U}_o := \mathcal{U} \cup \Big(\bigcup_{0\leq \ell\leq n} \mathrm{Spec}\,(B_\ell)\Big). \tag{4.8}$$

Proposition 4.1. *Assume that $\lambda \in \mathbb{C}\setminus\mathcal{U}_o$ and that $p_\Omega < p < q < q_\Omega$. Then, if $u \in \mathrm{Dom}\,(B_p)$ is such that $(\lambda I - \Delta)u \in L^q(\Omega, \Lambda^\ell)$, it follows that $u \in \mathrm{Dom}\,(B_q)$.*

Proof. Denote by $\tilde{f}$ the extension of $f := (\lambda I - \Delta)u \in L^q(\Omega,\Lambda^\ell)$ by zero in $\mathcal{M}\setminus\Omega$ and set

$$\eta := \big[(\Delta - \lambda I)^{-1}\tilde{f}\big]\Big|_{\Omega} \in L^q_2(\Omega,\Lambda^\ell). \tag{4.9}$$

Then the differential form

$$\begin{aligned} w \;\; := \;\; & \mathcal{S}_{\lambda,\ell}\Big[\big(-\tfrac{1}{2}I + M_{\lambda,\ell}\big)^{-1}(\nu\vee d\eta)\Big] \\ & + d\mathcal{S}_{\lambda,\ell-1}\Big\{\big(-\tfrac{1}{2}I + M_{\lambda,\ell-1}\big)^{-1}\Big(\nu\vee\eta - \nu\vee\mathcal{S}_{\lambda,\ell}\Big[\big(-\tfrac{1}{2}I + M_{\lambda,\ell}\big)^{-1}(\nu\vee d\eta)\Big]\Big)\Big\} \end{aligned} \tag{4.10}$$

satisfies, thanks to (4.4), (4.5), (4.7), and the fact that $d\delta = -\Delta + \delta d$,

$$\left\{\begin{array}{l} (\Delta - \lambda I)w = 0 \text{ in } \Omega, \\ w,\, d\delta w,\, \delta d w \in L^q(\Omega,\Lambda^\ell), \\ dw \in L^q(\Omega,\Lambda^{\ell+1}),\ \delta w \in L^q(\Omega,\Lambda^{\ell-1}), \\ \nu\vee w = \nu\vee\eta, \\ \nu\vee dw = \nu\vee d\eta. \end{array}\right. \tag{4.11}$$

It follows that $w - \eta \in \mathrm{Dom}\,(B_q) \hookrightarrow \mathrm{Dom}\,(B_p)$ and $(\lambda I - B_p)(w-\eta) = f$. Consequently, $u = (\lambda I - B_p)^{-1} f = w - \eta \in \mathrm{Dom}\,(B_q)$, as claimed. $\square$

Our last result in this section can, informally speaking, be regarded as a statement about the L^p-boundedness of the Riesz transforms $d\delta\Delta^{-1}$, $d\delta\Delta^{-1}$. Alternatively, it is a statement about the maximal regularity of $-\Delta$ relative to the decomposition $-\Delta = \delta d + d\delta$.

Proposition 4.2. *Assume that Ω is a Lipschitz subdomain of the manifold $\mathcal{M}$. If $p_\Omega < p < q_\Omega$ and $u \in \mathrm{Dom}\,(B_p)$ then*

$$\|d\delta u\|_{L^p(\Omega,\Lambda^\ell)} + \|\delta d u\|_{L^p(\Omega,\Lambda^\ell)} \leq C\|\Delta u\|_{L^p(\Omega,\Lambda^\ell)} \tag{4.12}$$

for some finite $C = C(\partial\Omega, p) > 0$.

Proof. Let the index p and the differential form u be as in the statement of the proposition and set $f := \Delta u \in L^p(\Omega,\Lambda^\ell)$. Applying G_p to this equality and relying on (3.19), leads to the conclusion that $u = G_p f + P_p u$. Hence, $d\delta u = d\delta G_p f$ and $\delta d u = \delta d G_p f$. Having justified this representation, the estimate (4.12) is a direct consequence of (3.18). $\square$

5 Main Lemma

For an arbitrary, fixed $\theta \in (0, \pi)$, consider the sector

$$\Sigma_\theta := \{z \in \mathbb{C} : \ |\arg z| < \pi - \theta\} \subset \mathbb{C} \tag{5.1}$$

and note that, generally speaking,

$$|\lambda a + b| \approx |\lambda| a + b, \quad \text{uniformly for} \quad \lambda \in \Sigma_\theta, \ a, b \geq 0. \tag{5.2}$$

In what follows, we shall work with the convention that

$$1 < p < \infty \Longrightarrow p^* := \frac{np}{n-1}. \tag{5.3}$$

Besides these conventions, below we collect a number of hypotheses which we will assume to be valid throughout this section.

Hypotheses. Consider an arbitrary $\theta \in (0, \pi)$, and arbitrary $\lambda \in \Sigma_\theta$ and set

$$t := \frac{1}{\sqrt{|\lambda|}} = |\lambda|^{-1/2}. \tag{5.4}$$

For a fixed $\ell \in \{0, 1, ..., n\}$, consider an arbitrary form

$$f \in C_0^\infty(\Omega, \Lambda^\ell) \hookrightarrow L^2(\Omega, \Lambda^\ell) \tag{5.5}$$

and define

$$u := (\lambda I - B_2)^{-1} f \in \mathrm{Dom}\,(B_2) \hookrightarrow L^2(\Omega, \Lambda^\ell). \tag{5.6}$$

Next, fix an arbitrary point $x \in \Omega$, an arbitrary sequence of functions $\{\eta_j\}_{j \geq 0}$ such that

$$\begin{aligned} &\eta_0 \in C_o^\infty(B(x, 2t)), \quad \eta_j \in C_o^\infty\Big(B(x, 2^{j+1}t) \setminus B(x, 2^{j-1}t)\Big) \\ &0 \leq \eta_j \leq 1, \qquad |\nabla \eta_j| \leq \frac{1}{2^{j-1}t} \qquad \sum_{j=0}^{\infty} \eta_j = 1, \end{aligned} \tag{5.7}$$

and decompose

$$f = \sum_{j=0}^{\infty} f_j, \qquad f_j := \eta_j f \in L^p(\Omega, \Lambda^\ell) \hookrightarrow L^2(\Omega, \Lambda^\ell), \tag{5.8}$$

$$u = \sum_{j=0}^{\infty} u_j, \qquad u_j := (\lambda I - B_2)^{-1} f_j \in \mathrm{Dom}\,(B_2). \tag{5.9}$$

Going further, assume that there exists

$$2 \le p < q_\Omega \tag{5.10}$$

with the property that for each $k \in \mathbb{N}$ there exists a finite, positive constant C_k depending only on k, θ, p, q, and the Lipschitz character of Ω such that

$$\begin{aligned}|\lambda|\Big[\int_{\Omega\cap B(x,t)}|u_j|^p\,dV\Big]^{1/p}+|\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|du_j|^q\,dV\Big]^{1/p}\\ +|\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|\delta u_j|^p\,dV\Big]^{1/p}\le\frac{C_k}{2^{kj}}\Big[\int_\Omega|f_j|^p\,dV\Big]^{1/p},\end{aligned} \tag{5.11}$$

for each $j \ge 0$, and such that

$$\begin{aligned}\Big[\int_{\Omega\cap B(x,t)}|\delta du_j|^p\,dV\Big]^{1/p}+\Big[\int_{\Omega\cap B(x,t)}|d\delta u_j|^p\,dV\Big]^{1/p}\\ \le\frac{C_k}{2^{kj}}\Big[\int_\Omega|f_j|^p\,dV\Big]^{1/p},\quad\forall\, j\ge 0.\end{aligned} \tag{5.12}$$

Lemma 5.1. *Granted the above conventions and assumptions, for each $k \in \mathbb{N}$ there exists a finite, positive constant C_k depending only on k, θ, p and the Lipschitz character of Ω such that, for each $j \ge 0$,*

$$\begin{aligned}|\lambda|\Big[\int_{\Omega\cap B(x,t)}|u_j|^{p^*}\,dV\Big]^{1/p^*}+|\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|du_j|^{p^*}\,dV\Big]^{1/p^*}\\ +|\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|\delta u_j|^{p^*}\,dV\Big]^{1/q^*}\le\frac{C_k}{2^{kj}}\Big[\int_\Omega|f_j|^{p^*}\,dV\Big]^{1/p^*},\end{aligned} \tag{5.13}$$

and, if in addition to the hypotheses made so far we also have

$$p^* < q_\Omega \tag{5.14}$$

then also

$$\begin{aligned}\Big[\int_{\Omega\cap B(x,t)}|\delta du_j|^{p^*}\,dV\Big]^{1/p^*}+\Big[\int_{\Omega\cap B(x,t)}|d\delta u_j|^{p^*}\,dV\Big]^{1/p^*}\\ \le\frac{C_k}{2^{kj}}\Big[\int_\Omega|f_j|^{p^*}\,dV\Big]^{1/p^*},\quad\forall\, j\ge 0.\end{aligned} \tag{5.15}$$

Proof. To prove this, we shall assume that (5.7)-(5.12) and proceed in a series of steps starting with

Step 1. *For each $j \geq 0$,*

$$\begin{aligned} & u_j \in L^2(\Omega, \Lambda^\ell), \quad du_j \in L^2(\Omega, \Lambda^{\ell+1}), \quad \delta u_j \in L^2(\Omega, \Lambda^{\ell-1}), \\ & \delta du_j,\, d\delta u_j \in L^2(\Omega, \Lambda^\ell), \quad \nu \vee u_j = 0, \quad \nu \vee du_j = 0, \end{aligned} \tag{5.16}$$

and

$$\lambda u_j - \Delta u_j = f_j \quad in \quad \Omega. \tag{5.17}$$

These follow from the definition of $u_j \in \mathrm{Dom}\,(B_2)$.

Step 2. *For any Lipschitz subdomain D of $\mathcal{M}$ there exists $2 < q_D \leq \infty$ with the following significance. For each $p \in [2, q_D)$ there exists $C > 0$ which depends exclusively on p and the Lipschitz character of D such that for any w satisfying*

$$\begin{aligned} & w \in L^p(D, \Lambda^\ell), \quad dw \in L^p(D, \Lambda^{\ell+1}), \quad \delta w \in L^p(D, \Lambda^{\ell-1}) \\ & \qquad \text{and such that } \ \nu \vee w = 0 \ \text{ on } \ \partial D, \end{aligned} \tag{5.18}$$

there holds

$$\begin{aligned} & R^{n\left(\frac{1}{p} - \frac{1}{p^*}\right)} \Big[\int_D |w|^{p^*}\, dV\Big]^{1/p^*} \\ & \quad \leq C \left\{ \Big[\int_D |w|^p\, dV\Big]^{1/p} + R\Big[\int_D |dw|^p\, dV\Big]^{1/p} + R\Big[\int_D |\delta w|^p\, dV\Big]^{1/p} \right\} \end{aligned} \tag{5.19}$$

where $R := \mathrm{diam}\,(D)$ and, as before, $p^* := np/(n-1)$.

To justify this, we first recall that, under the assumptions (5.18), the estimate

$$\|w\|_{L^p_{1/p}(D,\Lambda^\ell)} \leq C\Big[\|w\|_{L^p(D,\Lambda^\ell)} + \|dw\|_{L^p(D,\Lambda^{\ell+1})} + \|\delta w\|_{L^p(D,\Lambda^{\ell-1})}\Big] \tag{5.20}$$

has been established in [11], for a constant $C = C(\partial D, \mathrm{diam}\, D) > 0$ independent of w. Now (5.19) follows from this, Sobolev's embedding theorem to the effect that

$$L^p_{1/p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \qquad p^* = \frac{np}{n-1}, \tag{5.21}$$

and rescaling.

Step 3. *For each $k \in \mathbb{N}$ there exists $C_k > 0$ such that*

$$|\lambda|\Big[\int_{\Omega\cap B(x,t)}|u_j|^{p^*}\,dV\Big]^{1/p^*} \leq \frac{C_k}{2^{kj}}\,t^{n\left(\frac{1}{p^*}-\frac{1}{p}\right)}\Big[\int_{\Omega}|f_j|^p\,dV\Big]^{1/p} \tag{5.22}$$

and

$$|\lambda|\Big[\int_{\Omega\cap B(x,t)}|u_j|^{p^*}\,dV\Big]^{1/p^*} \leq \frac{C_k}{2^{kj}}\Big[\int_{\Omega}|f_j|^{p^*}\,dV\Big]^{1/p^*} \tag{5.23}$$

for each $j \geq 0$.

To prove this, fix a function

$$\zeta \in C^\infty_o(B(x,t)) \ \text{ with } \ \zeta \equiv 1 \text{ on } B(x,t/2) \text{ and } |\nabla\zeta| \leq C\,t^{-1} \tag{5.24}$$

and use (5.19) in the context when $D := \Omega \cap B(x,t)$ and $w := \zeta u_j$. That this applies is guaranteed by the fact that $\nu \vee (\zeta u_j) = 0$ on $\partial[\Omega \cap B(x,t)]$ and

$$\begin{aligned}
&\|\zeta u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^\ell)} \leq \|u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^\ell)},\\
&\|d(\zeta u_j)\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell+1})} \leq C\,t^{-1}\|u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^\ell)} + \|du_j\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell+1})},\\
&\|\delta(\zeta u_j)\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell-1})} \leq C\,t^{-1}\|u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^\ell)} + \|\delta u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell-1})}.
\end{aligned} \tag{5.25}$$

Also, the Lipschitz character of D is controlled by that of Ω and, hence, there is no loss of generality in assuming that $q_D = q_\Omega$. Thus, we may write

$$\begin{aligned}
t^{n\left(\frac{1}{p}-\frac{1}{p^*}\right)}&\Big[\int_{\Omega\cap B(x,t/2)}|u_j|^{p^*}\,dV\Big]^{1/p^*} \leq t^{n\left(\frac{1}{p}-\frac{1}{p^*}\right)}\Big[\int_{\Omega\cap B(x,t)}|\zeta u_j|^{p^*}\,dV\Big]^{1/p^*}\\
&\leq C\Big[\int_{\Omega\cap B(x,t)}|\zeta u_j|^p\,dV\Big]^{1/p} + C\,t\Big[\int_{\Omega\cap B(x,t)}|d(\zeta u_j)|^p\,dV\Big]^{1/p}\\
&\quad +C\,t\Big[\int_{\Omega\cap B(x,t)}|\delta(\zeta u_j)|^p\,dV\Big]^{1/p}\\
&\leq C\Big[\int_{\Omega\cap B(x,t)}|u_j|^p\,dV\Big]^{1/p} + \frac{C}{|\lambda|^{1/2}}\Big[\int_{\Omega\cap B(x,t)}|du_j|^p\,dV\Big]^{1/p}\\
&\quad +\frac{C}{|\lambda|^{1/2}}\Big[\int_{\Omega\cap B(x,t)}|\delta u_j|^p\,dV\Big]^{1/p}\\
&\leq \frac{C_k}{2^{kj}}|\lambda|^{-1}\Big[\int_{\Omega}|f_j|^p\,dV\Big]^{1/p},
\end{aligned} \tag{5.26}$$

where the last step is based on (5.11). This proves (5.22) with t replaced by $t/2$. This, however, is easily remedied by carrying out the same program with t replaced by $2t$ in (5.7), (5.24) and the definition of D. Going further, Hölder's inequality and the support condition on f_j gives

$$
\begin{aligned}
\Big[\int_{\Omega}|f_j|^p\,dV\Big]^{1/p} &= \Big[\int_{\Omega\cap B(x,2^{j+1}t)}|f_j|^p\,dV\Big]^{1/p} \\
&\leq C\,2^{jn}t^{n\left(\frac{1}{p}-\frac{1}{p^*}\right)}\Big[\int_{\Omega}|f_j|^{p^*}\,dV\Big]^{1/p^*}. \qquad (5.27)
\end{aligned}
$$

In concert with (5.26), this proves a version of (5.23) with t replaced by $t/2$ though, as before, this aspect is easily fixed.

Step 4. *For each $k\in\mathbb{N}$ there exists $C_k>0$ such that*

$$
\begin{aligned}
|\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|du_j|^{p^*}\,dV\Big]^{1/p^*} &+ |\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|\delta u_j|^{p^*}\,dV\Big]^{1/p^*} \\
&\leq \frac{C_k}{2^{kj}}\,t^{n\left(\frac{1}{p^*}-\frac{1}{p}\right)}\Big[\int_{\Omega}|f_j|^p\,dV\Big]^{1/p} \qquad (5.28)
\end{aligned}
$$

and

$$
\begin{aligned}
|\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|du_j|^{p^*}\,dV\Big]^{1/p^*} &+ |\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|\delta u_j|^{p^*}\,dV\Big]^{1/p^*} \\
&\leq \frac{C_k}{2^{kj}}\Big[\int_{\Omega}|f_j|^{p^*}\,dV\Big]^{1/p^*} \qquad (5.29)
\end{aligned}
$$

for each $j\geq 0$.

To justify these inequalities, pick a function ζ as in (5.24) and invoke (5.19) for $D:=\Omega\cap B(x,t)$ and $w:=\zeta\delta u_j$. Note that $\nu\vee(\zeta\delta u_j)=-\zeta\delta_\partial(\nu\vee u_j)=0$ on $\partial[\Omega\cap B(x,t)]$ and

$$
\begin{aligned}
&\|\zeta\delta u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell-1})}\leq\|\delta u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell-1})}, \\
&\|d(\zeta\delta u_j)\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell})}\leq C\,t^{-1}\|\delta u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell-1})}+\|d\delta u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell})}, \qquad (5.30) \\
&\|\delta(\zeta\delta u_j)\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell-2})}\leq C\,t^{-1}\|\delta u_j\|_{L^p(\Omega\cap B(x,t),\Lambda^{\ell-1})}.
\end{aligned}
$$

As before, we may assume that $q_D=q_\Omega$. Hence, we may estimate

$$
\begin{aligned}
t^{n\left(\frac{1}{p}-\frac{1}{p^*}\right)}\Big[\int_{\Omega\cap B(x,t/2)}|\delta u_j|^{p^*}\,dV\Big]^{1/p^*} &\leq t^{n\left(\frac{1}{p}-\frac{1}{p^*}\right)}\Big[\int_{\Omega\cap B(x,t)}|\zeta\delta u_j|^{p^*}\,dV\Big]^{1/p^*}\\
&\leq C\Big[\int_{\Omega\cap B(x,t)}|\zeta\delta u_j|^{p}\,dV\Big]^{1/p}+C\,t\Big[\int_{\Omega\cap B(x,t)}|d(\zeta\delta u_j)|^{p}\,dV\Big]^{1/p}\\
&\quad +C\,t\Big[\int_{\Omega\cap B(x,t)}|\delta(\zeta\delta u_j)|^{p}\,dV\Big]^{1/p}\\
&\leq C\Big[\int_{\Omega\cap B(x,t)}|\delta u_j|^{p}\,dV\Big]^{1/p}+\frac{C}{|\lambda|^{1/2}}\Big[\int_{\Omega\cap B(x,t)}|d\delta u_j|^{p}\,dV\Big]^{1/p}\\
&\leq \frac{C_k}{2^{kj}}|\lambda|^{-1}\Big[\int_{\Omega}|f_j|^{p}\,dV\Big]^{1/p}, \qquad (5.31)
\end{aligned}
$$

where we have utilized (5.11) and (5.12) in the last step. From this, we readily obtain

$$
|\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|\delta u_j|^{p^*}\,dV\Big]^{1/p^*}\leq \frac{C_k}{2^{kj}}\,t^{n\left(\frac{1}{p^*}-\frac{1}{p}\right)}\Big[\int_{\Omega}|f_j|^{p}\,dV\Big]^{1/p} \qquad (5.32)
$$

and, in a similar manner,

$$
|\lambda|^{1/2}\Big[\int_{\Omega\cap B(x,t)}|du_j|^{p^*}\,dV\Big]^{1/p^*}\leq \frac{C_k}{2^{kj}}\,t^{n\left(\frac{1}{p^*}-\frac{1}{p}\right)}\Big[\int_{\Omega}|f_j|^{p}\,dV\Big]^{1/p}. \qquad (5.33)
$$

Together, they prove (5.28). Finally, (5.29) follows from this and (5.27).

Step 5. *If $p^*<q_\Omega$ then for each $k\in\mathbb{N}$ there exists $C_k>0$ such that*

$$
\begin{aligned}
\Big[\int_{\Omega\cap B(x,t)}|\delta du_j|^{p^*}\,dV\Big]^{1/p^*}&+\Big[\int_{\Omega\cap B(x,t)}|d\delta u_j|^{p^*}\,dV\Big]^{1/p^*}\\
&\leq \frac{C_k}{2^{kj}}\,t^{n\left(\frac{1}{p^*}-\frac{1}{p}\right)}\Big[\int_{\Omega}|f_j|^{p}\,dV\Big]^{1/p} \qquad (5.34)
\end{aligned}
$$

and

$$
\begin{aligned}
\Big[\int_{\Omega\cap B(x,t)}|\delta du_j|^{p^*}\,dV\Big]^{1/p^*}&+\Big[\int_{\Omega\cap B(x,t)}|d\delta u_j|^{p^*}\,dV\Big]^{1/p^*}\\
&\leq \frac{C_k}{2^{kj}}\Big[\int_{\Omega}|f_j|^{p^*}\,dV\Big]^{1/p^*} \qquad (5.35)
\end{aligned}
$$

for each $j\geq 3$.

Assume that $j \geq 3$ and, once again, pick a function ζ as in (5.24). Thus,

$$\nu \vee (\zeta \delta du_j) = -\zeta \delta_\partial (\nu \vee du_j) = 0 \text{ on } \partial[\Omega \cap B(x,t)]. \tag{5.36}$$

Also, since $\zeta f_j = 0$ for $j \geq 3$,

$$\zeta \delta du_j = -\zeta \Delta u_j - \zeta d\delta u_j = -\zeta(\lambda\, u_j - f_j) - \zeta d\delta u_j = -\lambda\, \zeta u_j - \zeta d\delta u_j \tag{5.37}$$

and, hence,

$$d(\zeta \delta du_j) = -\lambda \mathcal{O}(|\nabla \zeta||u_j|) - \lambda \zeta du_j + \mathcal{O}(|\nabla \zeta||d\delta u_j|). \tag{5.38}$$

In particular,

$$\begin{aligned} \|d(\zeta \delta du_j)\|_{L^p(\Omega,\Lambda^{\ell+1})} \;\; &\leq \;\; C|\lambda| t^{-1} \|u_j\|_{L^p(\Omega,\Lambda^\ell)} + |\lambda| \|du_j\|_{L^p(\Omega,\Lambda^{\ell+1})} \\ &\quad + C\, t^{-1} \|d\delta u_j\|_{L^p(\Omega,\Lambda^\ell)}. \end{aligned} \tag{5.39}$$

Since also

$$\begin{aligned} &\|\zeta \delta du_j\|_{L^p(\Omega \cap B(x,t),\Lambda^\ell)} \leq \|\delta du_j\|_{L^p(\Omega \cap B(x,t),\Lambda^\ell)}, \\ &\|\delta(\zeta \delta du_j)\|_{L^p(\Omega \cap B(x,t),\Lambda^\ell)} \leq C\, t^{-1} \|\delta du_j\|_{L^p(\Omega \cap B(x,t),\Lambda^\ell)}, \end{aligned} \tag{5.40}$$

the estimate (5.19) is applicable to $D := \Omega \cap B(x,t)$ and $w := \zeta \delta du_j$ (assuming that that $q_D = q_\Omega$, which can be arranged). As a result, we have

$$\begin{aligned} t^{n\left(\frac{1}{p} - \frac{1}{p^*}\right)} \Big[\int_{\Omega \cap B(x,t/2)} |\delta du_j|^{p^*} \, dV\Big]^{1/p^*} &\leq t^{n\left(\frac{1}{p} - \frac{1}{p^*}\right)} \Big[\int_{\Omega \cap B(x,t)} |\zeta \delta du_j|^{p^*} \, dV\Big]^{1/p^*} \\ \leq C \Big[\int_{\Omega \cap B(x,t)} |\delta du_j|^p \, dV\Big]^{1/p} &+ C\, |\lambda| \Big[\int_{\Omega \cap B(x,t)} |u_j|^p \, dV\Big]^{1/p} \\ + C\, |\lambda|^{1/2} \Big[\int_{\Omega \cap B(x,t)} |du_j|^p \, dV &+ C\, \Big[\int_{\Omega \cap B(x,t)} |d\delta u_j|^p \, dV\Big]^{1/p} \\ \leq \frac{C_k}{2^{kj}} \Big[\int_\Omega |f_j|^p \, dV\Big]^{1/p}, & \end{aligned} \tag{5.41}$$

where the last step utilizes (5.12) and (5.11) In turn, from (5.41), Hölder's inequality and rescaling we readily obtain

$$\Big[\int_{\Omega\cap B(x,t)}|\delta du_j|^{p^*}\,dV\Big]^{1/p^*} \leq \frac{C_k}{2^{kj}}\,t^{n\left(\frac{1}{p^*}-\frac{1}{p}\right)}\Big[\int_{\Omega}|f_j|^p\,dV\Big]^{1/p}$$

$$\leq \frac{C_k}{2^{kj}}\Big[\int_{\Omega}|f_j|^{p^*}\,dV\Big]^{1/p^*}. \tag{5.42}$$

Going further, we write $d\delta u_j = -\Delta u_j + \delta du_j = f_j - \lambda u_j + \delta du_j$ and, consequently, $d\delta u_j = -\lambda u_j + \delta du_j$ on $\Omega\cap B(x,t)$ if $j\geq 3$. Hence, based on this, (5.42) and (5.22), we may estimate

$$\Big[\int_{\Omega\cap B(x,t)}|d\delta u_j|^{p^*}\,dV\Big]^{1/p^*} \leq |\lambda|\Big[\int_{\Omega\cap B(x,t)}|u_j|^{p^*}\,dV\Big]^{1/p^*}+\Big[\int_{\Omega\cap B(x,t)}|\delta du_j|^{p^*}\,dV\Big]^{1/p^*}$$

$$\leq \frac{C_k}{2^{kj}}\,t^{n\left(\frac{1}{p^*}-\frac{1}{p}\right)}\Big[\int_{\Omega}|f_j|^p\,dV\Big]^{1/p}$$

$$\leq \frac{C_k}{2^{kj}}\Big[\int_{\Omega}|f_j|^{p^*}\,dV\Big]^{1/p^*}. \tag{5.43}$$

Clearly, (5.42)-(5.43) prove (5.28)-(5.29).

Step 6. *Granted (5.4)-(5.12), for each $q\in(p,p^*]$ there exists $C=C(\partial\Omega,q)>0$ such that*

$$|\lambda|\|u\|_{L^q(\Omega,\Lambda^\ell)}\leq C\|f\|_{L^q(\Omega,\Lambda^\ell)} \tag{5.44}$$

and

$$|\lambda|^{1/2}\|du\|_{L^q(\Omega,\Lambda^\ell)}+|\lambda|^{1/2}\|\delta u\|_{L^q(\Omega,\Lambda^\ell)}\leq C\|f\|_{L^q(\Omega,\Lambda^\ell)}. \tag{5.45}$$

Given $q\in(p,p^*]$, select $\theta\in(0,1]$ such that $1/q=(1-\theta)/p+\theta/p^*$. From (5.11) and (5.23) we then obtain

$$|\lambda|\|u_j\|_{L^q(B(x,t)\cap\Omega,\Lambda^\ell)} \leq \Big[|\lambda|\|u_j\|_{L^p(B(x,t)\cap\Omega,\Lambda^\ell)}\Big]^{1-\theta}\Big[|\lambda|\|u_j\|_{L^{p^*}(B(x,t)\cap\Omega,\Lambda^\ell)}\Big]^{\theta}$$

$$\leq C_k\,2^{-kj}\,t^{n\left(\frac{\theta}{p^*}-\frac{\theta}{p}\right)}\,\|f_j\|_{L^p(\Omega,\Lambda^\ell)}$$

$$= C_k\,2^{-kj}\,t^{n\left(\frac{1}{q}-\frac{1}{p}\right)}\,\|f_j\|_{L^p(\Omega,\Lambda^\ell)}. \tag{5.46}$$

Now, with $⨍_E g\,dV := [\text{measure}\,(E)]^{-1}\int_E g\,dV$, and with M denoting the Hardy-Littlewood maximal operator, Fubini's Theorem and (5.46) allow us to write

$$
\begin{aligned}
|\lambda|\Big[\int_\Omega |u|^q\,dV\Big]^{1/q} &\leq C\Big[\int_\Omega\Big(\fint_{\Omega\cap B(x,t)}|u|^q\,dV\Big)dV_x\Big]^{1/q}\\
&= C|\lambda|\Big\{\int_\Omega\Big[\Big(\fint_{\Omega\cap B(x,t)}|u|^q\,dV\Big)^{1/q}\Big]^q dV_x\Big\}^{1/q}\\
&\leq C|\lambda|\Big\{\int_\Omega\Big[\sum_{j=0}^\infty\Big(\fint_{\Omega\cap B(x,t)}|u_j|^q\,dV\Big)^{1/q}\Big]^q dV_x\Big\}^{1/q}\\
&\leq C_k\Big\{\int_\Omega\Big[\sum_{j=0}^\infty 2^{-kj+jn/p}\Big(\fint_{\Omega\cap B(x,2^jt)}|f|^p\,dV\Big)^{1/p}\Big]^q dV_x\Big\}^{1/q}\\
&\leq C_k\Big(\sum_{j=0}^\infty 2^{-kj+jn/p}\Big)\|M(|f|^p)\|^{1/p}_{L^{q/p}(\Omega)}\\
&\leq C\|f\|_{L^q(\Omega)}, \qquad (5.47)
\end{aligned}
$$

if $k>1+n/p$. This proves (5.44).

The estimate (5.45) is then justified in a similar manner, by relying on (5.11) and (5.29).

Step 7. *The estimate (5.35) also holds if* $0\leq j\leq 3$.

It suffices to show that there exists $C=C(\partial\Omega,p)>0$ such that

$$
\|\delta du_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}+\|d\delta u_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}\leq C\,\|f_j\|_{L^{p^*}(\Omega,\Lambda^\ell)} \qquad (5.48)
$$

for each $j\geq 0$. To this end, we first note that the conclusion in Step 6 (with $q=p^*$) applied to u_j, f_j in place of u, f, yields

$$
|\lambda|\|u_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}\leq C\,\|f_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}, \quad \forall\, j\geq 0. \qquad (5.49)
$$

Next, recall that $u_j\in\mathrm{Dom}\,(B_2)$ and $(\lambda I-\Delta)u_j=f_j\in C^\infty_0(\Omega,\Lambda^\ell)\hookrightarrow L^{p^*}(\Omega,\Lambda^\ell)$. Since we are assuming that $2<p^*<q_\Omega$, Proposition 4.1 guarantees that $u_j\in\mathrm{Dom}\,(B_{p^*})$. Consequently, (4.12) and (5.49) allow us to write

$$
\begin{aligned}
\|d\delta u_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}+\|\delta du_j\|_{L^{p^*}(\Omega,\Lambda^\ell)} &\leq C\|\Delta u_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}\\
&= C\|\lambda u_j-f_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}\leq C|\lambda|\|u_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}+C\|f_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}\\
&\leq C\|f_j\|_{L^{p^*}(\Omega,\Lambda^\ell)}, \qquad \forall\, j\geq 0, \qquad (5.50)
\end{aligned}
$$

for some finite $C=C(\partial\Omega,p)>0$. Thus, (5.48) is proved.

Step 8. *Proof of (5.13), (5.15).*

Note that (5.13) is a consequence of (5.23) and (5.29). Finally, (5.35) takes care of the case $j \geq 3$ of (5.15), whereas the case $0 \leq j \leq 3$ is contained in Step 7.

This finishes the proof of Lemma 5.1. $\square$

6 Resolvent estimates

In this section we shall make use of Lemma 5.1 in order to prove resolvent estimates for the Hodge-Laplacian.

Theorem 6.1. *Let $\Omega \subset \mathcal{M}$ be a Lipschitz domain, and fix $\ell \in \{0, 1, ..., n\}$ and $\theta \in (0, \pi)$. Then for each $\lambda \in \Sigma_\theta$ and each*

$$p \in (p_\Omega, q_\Omega) \tag{6.1}$$

the (unbounded) operator

$$\lambda I - B_p : \mathrm{Dom}\,(B_p) \subset L^p(\Omega, \Lambda^\ell) \longrightarrow L^p(\Omega, \Lambda^\ell) \tag{6.2}$$

has a bounded inverse. Furthermore, there exists $C = C(\partial\Omega, \theta, p) > 0$ such that

$$\|(\lambda I - B_p)^{-1} f\|_{L^p(\Omega,\Lambda^\ell)} \leq C\,|\lambda|^{-1}\|f\|_{L^p(\Omega,\Lambda^\ell)}, \tag{6.3}$$

$$\|d(\lambda I - B_p)^{-1} f\|_{L^p(\Omega,\Lambda^{\ell-1})} + \|\delta(\lambda I - B_p)^{-1} f\|_{L^p(\Omega,\Lambda^{\ell+1})} \leq C\,|\lambda|^{-1/2}\|f\|_{L^p(\Omega,\Lambda^\ell)}, \tag{6.4}$$

$$\|d\delta(\lambda I - B_p)^{-1} f\|_{L^p(\Omega,\Lambda^\ell)} + \|\delta d(\lambda I - B_p)^{-1} f\|_{L^p(\Omega,\Lambda^\ell)} \leq C\,\|f\|_{L^p(\Omega,\Lambda^\ell)}, \tag{6.5}$$

for any $\lambda \in \Sigma_\theta$ and any $f \in L^p(\Omega, \Lambda^\ell)$.

Proof. Consider first the case when $p \in (2, q_\Omega)$. In this scenario, for an arbitrary $\lambda \in \Sigma_\theta$, the fact that the operator (6.2) is one-to-one follows trivially from the corresponding statement for $p = 2$ (dealt with in §3). To see that this operator is also onto, let $f \in L^p(\Omega, \Lambda^\ell) \hookrightarrow L^p(\Omega, \Lambda^\ell)$ and consider $u := (\lambda I - B_2)^{-1} f \in \mathrm{Dom}\,(B_2)$. Thanks to Proposition 4.1, we have that $u \in \mathrm{Dom}\,(B_p)$ and $(\lambda I - B_p)u = f$ which proves that the operator (6.2) is indeed onto.

Turning our attention to (6.3)-(6.4), we note that it suffices to prove these estimates for an arbitrary $f \in C^\infty_0(\Omega, \Lambda^\ell)$. With the notation and conventions introduces in §5, these are going to be consequences of (5.44) and (5.45), provided we show that the index q appearing there can be chosen arbitrarily in $(2, q_\Omega)$. In turn, by virtue of the inductive bootstrap argument in Lemma 5.1, this latter condition will hold as soon as we prove that (5.11)-(5.12) are valid for the choice $p = 2$.

With this goal in mind, we start by pairing both sides of (5.17) with $\bar{u}_j$ in the L^2-sense. After integrating by parts, we eventually obtain

$$\lambda \int_\Omega |u_j|^2 \, dV + \int_\Omega |du_j|^2 \, dV + \int_\Omega |\delta u_j|^2 \, dV = \int_\Omega \langle f_j, \bar{u}_j \rangle \, dV. \tag{6.6}$$

From this we may further deduce, based on (5.2) and the Cauchy-Schwarz inequality, that

$$|\lambda| \|u_j\|_{L^2(\Omega,\Lambda^\ell)} + |\lambda|^{1/2} \|du_j\|_{L^2(\Omega,\Lambda^{\ell+1})} + |\lambda|^{1/2} \|\delta u_j\|_{L^2(\Omega,\Lambda^{\ell-1})} \leq C \|f_j\|_{L^2(\Omega,\Lambda^\ell)}. \tag{6.7}$$

Next, with t retaining the same significance as before, i.e. $t := |\lambda|^{-1/2}$, pick a new family of functions $\{\xi_j\}_{j \geq 3}$ such that

$$\begin{aligned} &\xi_j \in C_o^\infty(B(x, 2^{j-2}t)), \quad \xi_j \equiv 1 \text{ on } B(x, 2^{j-3}t), \\ &0 \leq \xi_j \leq 1, \qquad |\nabla \xi_j| \leq \frac{C}{2^j t}, \quad \text{for each } j \geq 3. \end{aligned} \tag{6.8}$$

Taking the L^2-pairing of $\xi^2 \bar{u}_j$ with both sides of (5.17) and keeping in mind that $\xi_j \eta_j = 0$ for each $j \geq 3$ we may write, based on integrations by parts that

$$\begin{aligned} \lambda \int_\Omega \xi_j^2 |u_j|^2 \, dV &+ \int_\Omega \xi_j^2 |du_j|^2 \, dV + \int_\Omega \xi_j^2 |\delta u_j|^2 \, dV \\ &= \int_\Omega \mathcal{O}\Big(|\nabla \xi_j| |u_j| \Big[|\xi_j| |du_j| + |\xi_j| |\delta u_j| \Big] \Big) \, dV. \end{aligned} \tag{6.9}$$

From this, (5.2) and Cauchy-Schwarz's inequality, we then obtain that, for $j \geq 3$,

$$\begin{aligned} |\lambda| \int_\Omega \xi_j^2 |u_j|^2 \, dV &+ \int_\Omega \xi_j^2 |du_j|^2 \, dV + \int_\Omega \xi_j^2 |\delta u_j|^2 \, dV \\ &\leq C \int_\Omega |\nabla \xi_j| |u_j| \Big[|\xi_j| |du_j| + |\xi_j| |\delta u_j| \Big] \, dV \\ &\leq C \|\nabla \xi_j\|_{L^\infty} \|u_j\|_{L^2(\operatorname{supp} \xi_j \cap \Omega, \Lambda^\ell)} \Big(\|\xi_j \, du_j\|_{L^2(\Omega,\Lambda^{\ell+1})} + \|\xi_j \, \delta u_j\|_{L^2(\Omega,\Lambda^{\ell-1})} \Big) \end{aligned} \tag{6.10}$$

which, via a standard trick that allows us to absorb the terms in the round parentheses in the left-most side, further gives

$$\begin{aligned} |\lambda| \int_\Omega \xi_j^2 |u_j|^2 \, dV &+ \int_\Omega \xi_j^2 |du_j|^2 \, dV + \int_\Omega \xi_j^2 |\delta u_j|^2 \, dV \\ &\leq C \|\nabla \xi_j\|_{L^\infty}^2 \|u_j\|_{L^2(\operatorname{supp} \xi_j \cap \Omega, \Lambda^\ell)}^2 \leq C \, 2^{-2j} |\lambda| \|u_j\|_{L^2(\operatorname{supp} \xi_j \cap \Omega, \Lambda^\ell)}^2 \end{aligned} \tag{6.11}$$

for each $j \geq 3$. Thus,

$$|\lambda| \int_{\Omega \cap B(x,2^{j-3}t)} |u_j|^2 \, dV + \int_{\Omega \cap B(x,2^{j-3}t)} |du_j|^2 \, dV + \int_{\Omega \cap B(x,2^{j-3}t)} |\delta u_j|^2 \, dV$$
$$\leq C\, 2^{-2j} |\lambda| \int_{\Omega \cap B(x,2^{j-2}t)} |u_j|^2 \, dV, \qquad \forall\, j \geq 3. \tag{6.12}$$

Given an arbitrary positive integer k and assuming that $j \geq k+3$ the procedure leading up to (6.12) can be iterated k times (choosing, at each step, a new family satisfying properties similar to (6.8) but on progressively smaller balls), yielding

$$|\lambda| \int_{\Omega \cap B(x,2^{j-3-k}t)} |u_j|^2 \, dV + \int_{\Omega \cap B(x,2^{j-3-k}t)} |du_j|^2 \, dV + \int_{\Omega \cap B(x,2^{j-3-k}t)} |\delta u_j|^2 \, dV$$
$$\leq C_k\, 2^{-(k+2)j} |\lambda| \int_{\Omega \cap B(x,2^{j-2-k}t)} |u_j|^2 \, dV, \qquad \forall\, j \geq k+3. \tag{6.13}$$

We are now ready to prove the case $p = 2$ of (5.11). In other words, we shall show that for any $k \in \mathbb{N}$ there exists a finite constant $C_k > 0$ such that

$$|\lambda| \int_{\Omega \cap B(x,t)} |u_j|^2 \, dV + \int_{\Omega \cap B(x,t)} |du_j|^2 \, dV + \int_{\Omega \cap B(x,t)} |\delta u_j|^2 \, dV$$
$$\leq \frac{C_k}{2^{kj}|\lambda|} \int_{\Omega} |f_j|^2 \, dV, \qquad \forall\, j \geq 0. \tag{6.14}$$

Indeed, for j large this follows from (6.13), whereas for j small (5.11) is a direct consequence of (6.7).

There remains to prove the $p = 2$ version of (5.12), a task to which we now turn. In fact, we aim at showing that

$$\int_{B(x,t/2) \cap \Omega} \Big\{ |d\delta u_j|^2 + |\delta d u_j|^2 \Big\} \, dV \leq C_k 2^{-kj} \int_{\Omega} |f_j|^2 \, dV, \tag{6.15}$$

which corresponds to (5.12) written for $p = 2$ and $t/2$ in place of t (the latter condition being just a minor technicality, easily addressed via rescaling). In turn, if ζ is as in (5.24), (6.15) will be a simple consequence of the estimate

$$\int_{B(x,t) \cap \Omega} \Big\{ |d(\zeta \delta u_j)|^2 + |\delta(\zeta d u_j)|^2 \Big\} \, dV \leq C_k 2^{-kj} \int_{\Omega} |f_j|^2 \, dV \tag{6.16}$$

which, so we claim, is valid for each $j \geq 0$. In order to justify (6.16), we shall first establish the estimate

$$\int_{B(x,t)\cap\Omega} |\zeta\Delta u_j|^2\, dV \le C_k 2^{-kj} \int_\Omega |f_j|^2\, dV, \qquad \forall\, j \ge 0. \tag{6.17}$$

To prove this, we first note that since $\Delta u_j = \lambda u_j - f_j$ in Ω for every j, then

$$\|\Delta u_j\|_{L^2(\Omega,\Lambda^\ell)} \le |\lambda| \|u_j\|_{L^2(\Omega,\Lambda^\ell)} + \|f_j\|_{L^2(\Omega,\Lambda^\ell)} \le C\|f_j\|_{L^2(\Omega,\Lambda^\ell)}, \tag{6.18}$$

by (6.7). As this implies (6.17) for small j's, we can assume for the remainder of the proof that $j \ge 2$. In particular, $f_j \equiv 0$ on $B(x,t)$. Next, multiply by ζ both sides of the equality $\Delta u_j = \lambda u_j - f_j$ to get $\zeta\Delta u_j = \lambda\zeta u_j$ and write

$$\begin{aligned}\int_{B(x,t)\cap\Omega} |\zeta\Delta u_j|^2\, dV &= |\lambda| \int_{B(x,t)\cap\Omega} |\zeta u_j|^2\, dV \\ &\le |\lambda| \int_{B(x,t)\cap\Omega} |u_j|^2\, dV \le C_k 2^{-kj} \int_\Omega |f_j|^2\, dV \end{aligned} \tag{6.19}$$

where in the last step we have used (6.14). This finishes the proof of (6.17).

To continue, write

$$-\zeta\Delta u_j = d(\zeta\delta u_j) + \delta(\zeta du_j) + \mathcal{O}\Big(|\nabla\zeta|\big[|du_j| + |\delta u_j|\big]\Big) \tag{6.20}$$

so that

$$\begin{aligned}&\int_{B(x,t)\cap\Omega} |d(\zeta\delta u_j) + \delta(\zeta du_j)|^2\, dV \\ &\quad = \int_{B(x,t)\cap\Omega} |\zeta\Delta u_j|^2\, dV + \int_{B(x,t)\cap\Omega} \mathcal{O}\Big(|\nabla\zeta|^2\big[|du_j|^2 + |\delta u_j|^2\big]\Big) dV \\ &\quad \le C_k 2^{-kj} \int_\Omega |f_j|^2\, dV + |\lambda| \int_{B(x,t)\cap\Omega} \big[|du_j|^2 + |\delta u_j|^2\big]\, dV \\ &\quad \le C_k 2^{-kj} \int_\Omega |f_j|^2\, dV \end{aligned} \tag{6.21}$$

by (6.17), (5.24) and (6.14). On the other hand,

$$|d(\zeta\delta u_j)|^2 + |\delta(\zeta du_j)|^2 = |d(\zeta\delta u_j) + \delta(\zeta du_j)|^2 - 2\,\mathrm{Re}\,\langle d(\zeta\delta u_j)\,,\,\delta(\zeta du_j)\rangle \tag{6.22}$$

and, via an integration by parts,

$$\int_{B(x,t)\cap\Omega} \langle d(\zeta\delta u_j)\,,\,\delta(\zeta du_j)\rangle\, dV = 0 \tag{6.23}$$

since $d^2 = 0$ and $\nu \vee (\zeta du_j) = \zeta(\nu \vee du_j) = 0$ on $\partial[\Omega \cap B(x,t)]$. Thus, all in all, (6.16) is a consequence of (6.22), (6.21) and (6.23), and this finishes the proof of (5.12) when $p = 2$. In turn, as explained earlier, this concludes the proof of (6.3)-(6.4) in the case when $2 < p < q_\Omega$.

As regards (6.5), we may invoke Proposition 4.2, the fact that $\Delta(\lambda I - B_p)^{-1} f = \lambda(\lambda I - B_p)^{-1} f - f$ and (6.3), in order to justify it in the case when $2 < p < q_\Omega$.

Finally, the case when $p_\Omega < p < 2$ follows from what we have proved so far and duality; cf. Proposition 3.3. Since the case $p = 2$ is implicit in the above analysis, this finishes the proof of Theorem 6.1. □

7 A_p, B_p and C_p generate analytic semigroups

We start we the case of the Hodge-Laplacian, for which we have:

Theorem 7.1. *If $\Omega \subset \mathcal{M}$ is a Lipschitz domain and $0 \leq \ell \leq n$, then the operator B_p generates an analytic semigroup in $L^p(\Omega, \Lambda^\ell)$ for each $p \in (p_\Omega, q_\Omega)$.*

More specifically, for each $\theta \in (0, \pi)$ there exists an analytic map

$$T : \Sigma_\theta \longrightarrow \mathcal{L}\Big(L^p(\Omega, \Lambda^\ell), L^p(\Omega, \Lambda^\ell)\Big) \tag{7.1}$$

such that the following hold:

$$\lim_{\substack{z \to 0 \\ z \in \Sigma_\theta}} T(z) f = f \text{ in } L^p(\Omega, \Lambda^\ell), \ \forall f \in L^p(\Omega, \Lambda^\ell), \tag{7.2}$$

$$T(z_1 + z_2) = T(z_1) T(z_2), \qquad \forall z_1, z_2 \in \Sigma_\theta, \tag{7.3}$$

$$\mathrm{Dom}\,(B_p) = \left\{ u \in L^p(\Omega, \Lambda^\ell) : \lim_{t \to 0^+} \frac{T(t)u - u}{t} \text{ exists in } L^p(\Omega, \Lambda^\ell) \right\}, \tag{7.4}$$

$$B_p u = \lim_{t \to 0^+} \frac{T(t)u - u}{t} \text{ for each } u \in \mathrm{Dom}\,(B_p). \tag{7.5}$$

Proof. This follows from Theorem 6.1 and the classical Hille-Yoshida theory; cf., e.g., [14]. □

As is customary, we shall set

$$e^{tB_p} := T(t), \qquad t > 0. \tag{7.6}$$

Corollary 7.2. *Under the hypotheses of Theorem 7.1, for each $t > 0$,*

$$\Big(e^{tB_p}\Big)^* = e^{tB_{p'}}, \qquad 1/p + 1/p' = 1, \tag{7.7}$$

and

$$\mathbb{P}_p\, e^{tB_p} = e^{tB_p}\, \mathbb{P}_p, \qquad \mathbb{Q}_p\, e^{tB_p} = e^{tB_p}\, \mathbb{Q}_p. \tag{7.8}$$

Proof. This is an immediate consequence of Theorem 7.1, Lemma 3.7 and (ii) in Lemma 3.13. □

Theorem 7.3. *For each Lipschitz domain $\Omega \subset \mathcal{M}$, $0 \leq \ell \leq n$, and $p \in (p_\Omega, q_\Omega)$, the Stokes operator A_p generates an analytic semigroup $\left(e^{tA_p}\right)_{t>0}$ on the space $X^p(\Omega, \Lambda^\ell)$.*

Furthermore, for each $t > 0$

$$\left(e^{tA_p}\right)^* = e^{tA_{p'}}, \qquad 1/p + 1/p' = 1, \tag{7.9}$$

and

$$\mathbb{P}_p\, e^{tB_p} = e^{tA_p}\, \mathbb{P}_p \quad \text{on} \quad L^p(\Omega, \Lambda^\ell), \tag{7.10}$$

whenever $p \in (p_\Omega, q_\Omega)$.

Proof. Fix some $\theta \in (0, \pi)$ and assume that $p \in (p_\Omega, q_\Omega)$. Using Lemma 3.10 and Theorem 6.1, for each $f \in X^p(\Omega, \Lambda^\ell)$ we may then write

$$\begin{aligned} \|(\lambda I - A_p)^{-1} f\|_{L^p(\Omega,\Lambda^\ell)} &= \|(\lambda I - A_p)^{-1} \mathbb{P}_p f\|_{L^p(\Omega,\Lambda^\ell)} \\ &= \|\mathbb{P}_p (\lambda I - B_p)^{-1} f\|_{L^p(\Omega,\Lambda^\ell)} \\ &\leq C|\lambda|^{-1} \|f\|_{L^p(\Omega,\Lambda^\ell)}, \end{aligned} \tag{7.11}$$

uniformly for $\lambda \in \Sigma_\theta$. Consequently, A_p generates an analytic semigroup on $X^p(\Omega, \Lambda^\ell)$ whenever $p \in (p_\Omega, q_\Omega)$.

Finally, (7.9) and (7.10) follow readily from this, Lemma 3.8 and Lemma 3.14. □

In a similar fashion, one can prove the following.

Theorem 7.4. *If $\Omega \subset \mathcal{M}$ is a Lipschitz domain, $0 \leq \ell \leq n$, and $p \in (p_\Omega, q_\Omega)$, the Maxwell operator C_p generates an analytic semigroup $\left(e^{tC_p}\right)_{t>0}$ on the space $Z^p(\Omega, \Lambda^\ell)$. Moreover, for each $t > 0$*

$$\left(e^{tC_p}\right)^* = e^{tC_{p'}}, \qquad 1/p + 1/p' = 1, \tag{7.12}$$

and

$$\mathbb{Q}_p\, e^{tB_p} = e^{tC_p}\, \mathbb{Q}_p \quad \text{on} \quad L^p(\Omega, \Lambda^\ell), \tag{7.13}$$

for each $p \in (p_\Omega, q_\Omega)$.

Theorem 7.5. *Fix $0 \leq \ell \leq n$ and suppose that $\Omega \subset \mathcal{M}$ is a Lipschitz domain for which $b_\ell = 0$. Then for each $p \in (p_\Omega, q_\Omega)$, the analytic semigroups generated by the operators A_p, B_p and C_p, respectively, on $L^p(\Omega, \Lambda^\ell)$, $X^p(\Omega, \Lambda^\ell)$ and $Z^p(\Omega, \Lambda^\ell)$ are bounded.*

Proof. This follows from Theorems 7.1, 7.3 and 7.4, given that under the current topological assumptions the operators A_p, B_p and C_p are invertible. □

In closing, we would like to point out that, as an obvious corollary of what we have proved so far, similar results are valid for the Hodge duals of the operators A_p, B_p and C_p (i.e., for $*A_p*$, $*B_p*$ and $*C_p*$). For example, corresponding to the Hodge dual of B_p, $-\Delta$ defined as an unbounded operator on $L^p(\Omega, \Lambda^\ell)$ with domain

$$\{u \in \mathcal{D}^p_\ell(\Omega; d) \cap \mathcal{D}^p_\ell(\Omega; \delta) : \, du \in \mathcal{D}^p_{\ell+1}(\Omega; \delta), \; \delta u \in \mathcal{D}^p_{\ell-1}(\Omega; d), \, \nu \wedge u = 0, \, \nu \wedge \delta u = 0\}$$

generates an analytic semigroup whenever $p_\Omega < p < q_\Omega$. We leave the details for the remaining operators to the interested reader.

References

[1] S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math., 15 (1962), 119–147.

[2] T. Akiyama, H. Kasai, Y. Shibata and M. Tsutsumi, *On a resolvent estimate of a system of Laplace operators with perfect wall condition*, Funkcial. Ekvac., 47 (2004), no. 3, 361–394.

[3] P.E. Conner, *The Neumann's problem for differential forms on Riemannian manifolds*, Mem. Amer. Math. Soc., No. 20, 1956.

[4] P. Deuring, *L^p-theory for the Stokes system in 3D domains with conical boundary points*, Indiana Univ. Math. J., 47 (1998), 11–47.

[5] T.G. Cowling, *Magnetohydrodynamics*, Interscience Tracts on Physics and Astronomy, No. 4 Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London 1957.

[6] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer-Verlag, Berlin, 1988.

[7] E. Fabes, O. Mendez and M. Mitrea, *Potential operators on Besov spaces and the Poisson equation with Dirichlet and Neumann boundary conditions on Lipschitz domains*, Journal of Functional Analysis, Vol. 159, No. 2 (1998), 323–368.

[8] A. Jonsson and H. Wallin, *Function Spaces on Subsets of* $\mathbb{R}^n$, Harwood Academic, New York, 1984.

[9] L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, Oxford-London-New York-Paris; Addison-Wesley Publishing Co., Inc., Reading, Mass., 1960.

[10] T. Miyakawa, *The* L^p *approach to the Navier-Stokes equations with the Neumann boundary condition*, Hiroshima Math. J., 10 (1980), no. 3, 517–537.

[11] D. Mitrea, M. Mitrea and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Memoirs of the Amer. Math. Soc., Vol. 150 No. 713, 2001.

[12] D. Mitrea and M. Mitrea, *Finite energy solutions of Maxwell's equations and constructive Hodge decompositions on nonsmooth Riemannian manifolds*, J. Funct. Anal., 190 (2002), 339–417.

[13] M. Mitrea, *Sharp Hodge decompositions, Maxwell's equations, and vector Poisson problems on nonsmooth, three-dimensional Riemannian manifolds*, Duke Math. J., 125 (2004), 467–547.

[14] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York, 1983.

[15] M.E. Taylor, *Partial Differential Equations*, Springer-Verlag, New York, 1996.

[16] M.E. Taylor, *Incompressible fluid flows on rough domains*, pp. 320–334 in Semigroups of Operators: Theory and Applications, Progr. Nonlinear Differential Equations Appl., Vol. 42, Birkhäuser, Basel, 2000.

Marius Mitrea
Department of Mathematics
University of Missouri-Columbia
Mathematical Sciences Building
Columbia, MO 65211, USA

e-mail: marius@math.missouri.edu

Sylvie Monniaux
LATP - UMR 6632
Faculté des Sciences
de Saint-Jérôme - Case Cour A
Université Aix-Marseille 3
F-13397 Marseille Cédex 20, France

e-mail: sylvie.monniaux@univ.u-3mrs.fr